Tutorial 9

Exercise 1: Prove in detail that $f \in O(g)$ and $g \in \Theta(f)$ holds for the following functions:

$$f(n) = 5n^3 + 2n^2 - 9n + 13 \qquad\qquad g(n) = n^3$$

Deduce from this that also $g \in O(f)$, $g \in \Omega(f)$, $f \in \Theta(g)$, $g \in \Theta(f)$, $g \notin o(f)$, and $g \notin \omega(f)$.

Exercise 2: Order the following functions according to their asymptotic growth, i.e., order them into a sequence $g_1, g_2, \ldots, g_{15}$, where $g_1 \in O(g_2)$, $g_2 \in O(g_3)$, $\ldots$, $g_{14} \in O(g_{15})$. Describe also, for which pairs of functions g_i and g_{i+1} in this sequence $g_i \in \Theta(g_{i+1})$ a pro které $g_i \in o(g_{i+1})$ holds.

$$\begin{array}{ccccc} n^2 & 2^n & n & \log_2 n & n^n \\ n! & \log_2(n^2) & (\log_2 n)^2 & n^3 & \sqrt{n} \\ 2^{2^n} & 10^n & n^{1000} & \sqrt[3]{n} & n \log_2 n \end{array}$$

Exercise 3: Determine for the following triples of functions f_1, f_2, f_3, which relations of the form $f_i \in O(f_j)$, $f_i \in \Omega(f_j)$, $f_i \in \Theta(f_j)$, $f_i \in o(f_j)$, and $f_i \in \omega(f_j)$ hold and which do not.

a) $f_1(n) = 3n^2 + 5n - 1, \quad f_2(n) = 2n^3 - 15n - 183, \quad f_3(n) = (n+1)(n-1)$
b) $f_1(n) = 4n^2 + n^2 \log_2 n, \quad f_2(n) = \log_2^5 n, \quad f_3(n) = 17n + 3$
c) $f_1(n) = n\sqrt[5]{n}, \quad f_2(n) = n, \quad f_3(n) = \sqrt{n}$
d) $f_1(n) = 2^n, \quad f_2(n) = n^{1024}, \quad f_3(n) = n!$
e) $f_1(n) = 2^n, \quad f_2(n) = n^n, \quad f_3(n) = n!$
f) $f_1(n) = 2^n, \quad f_2(n) = n^n, \quad f_3(n) = n^{\log_2 n}$
g) $f_1(n) = 10^n, \quad f_2(n) = 2^n, \quad f_3(n) = 2^{2^n}$
h) $f_1(n) = \log_{10}(n^2), \quad f_2(n) = \log_2 n, \quad f_3(n) = \log_2(n^2)$
i) $f_1(n) = n + \sqrt{n} \cdot \log_2 n, \quad f_2(n) = n \cdot \log_2 n, \quad f_3(n) = \sqrt{n} \cdot \log_2^2 n$
j) $f_1(n) = 2^n, \quad f_2(n) = 2^{\sqrt{n}}, \quad f_3(n) = n!$
k) $f_1(n) = n/2048, \quad f_2(n) = \sqrt{n} \cdot 3n, \quad f_3(n) = n + n \cdot \log_2 n$
l) $f_1(n) = (\log_2 n)^n, \quad f_2(n) = n^n, \quad f_3(n) = 10^{\sqrt{n}}$

Exercise 4: Determine as precisely as possible the time and space complexity of Algorithm 1. You can assume that value n represents the number of elements in array A, and that this array is indexed from zero.

Exercise 5: Determine as precisely as possible the time and space complexity of Algorithm 2 (recall this algorithm from the previous tutorial).

(You can assume that value n represents the number of elements in array A, that this array is indexed from zero, and that x is a value of searched element.)

Algorithm 1: Selection sort

```
Selection-Sort $(A, n)$:
    $i := n - 1$
    while $i > 0$ do
        $k := 0$
        for $j := 1$ to $i$ do
            if $A[k] < A[j]$ then
                $k := j$
        $x := A[k]$;  $A[k] := A[i]$;  $A[i] := x$
        $i := i - 1$
```

Algorithm 2: Binary search

```
BSearch $(x, A, n)$:
    $\ell := 0$
    $r := n$
    while $\ell < r$ do
        $k := \lfloor (\ell + r) / 2 \rfloor$
        if $A[k] < x$ then
            $\ell := k + 1$
        else
            $r := k$
    if $\ell < n$ and $A[\ell] = x$ then
        return $\ell$
    return NotFound
```

Exercise 6: By pseudocode describe an arbitrary algorithm for solving the following problem, and estimate its time and space complexity as accurately as possible. (What is an appropriate measure of the size of an input in this problem?)

INPUT: Matrices A, B with integer elements.

OUTPUT: Matrix $A \cdot B$.

Remark: You don't have to deal with input and output in your algorithm.

Do not assume that matrices A and B must be square matrices. However, you can assume that sizes of both matrices are such that it is possible to multiply the matrices, i.e., that matrix A is of size $m \times n$ a matrix B is of size $n \times p$, where m, n, and p are some natural numbers.

Exercise 7: Design an algorithm solving the following problem:

INPUT: A number n and a sequence of numbers $a_1, a_2, \ldots, a_n$, where for each $i = 1, 2, \ldots, n$ is $a_i \in \{1, 2, \ldots, n\}$.

QUESTION: Does the sequence $a_1, a_2, \ldots, a_n$ contain every $x \in \{1, 2, \ldots, n\}$ exactly once?

Analyze the time complexity of your algorithm. If it is greater than $O(n)$, try to design an algorithm with the time complexity $O(n)$.