Tutorial 1

Exercise 1: For each of the following languages, give an example of 5 words belonging to the language, and an example of 5 words that do not belong to the language.

a) $L_1 = \{w \in \{0,1\}^* \mid \text{the length of word } w \text{ is less than 5}\}$

Solution: Examples of words that belong to language L_1: ε, 0, 1, 00, 01, ...
Examples of words that do not belong to language L_1: 00000, 00001, 00010, 000000, 1111111, ...

b) $L_2 = \{w \in \{a,b\}^* \mid \text{the number of occurrences of symbol b in word } w \text{ is even}\}$

Solution: Examples of words that belong to language L_2: ε, a, aa, bb, aaa, abb, ...
Examples of words that do not belong to language L_2: b, ab, ba, aab, aba, ...

c) $L_3 = \{w \in \{0,1\}^* \mid \text{in } w \text{ is every 0 (directly) followed by 1}\}$

Solution: Examples of words that belong to language L_3: ε, 1, 01, 11, 101101, ...
Examples of words that do not belong to language L_3: 0, 10, 001, 010, 1010, ...

d) $L_4 = \{w \in \{0,1\}^* \mid w \text{ begins and ends with the same symbol}\}$

Solution: Examples of words that belong to language L_4: 0, 1, 00, 11, 000, 010, ...
Examples of words that do not belong to language L_4: ε, 01, 10, 001, 011, ...

e) $L_5 = \{w \in \{a,b\}^* \mid w \text{ contains as a subword the sequence abb}\}$

Solution: Examples of words that belong to language L_5: abb, aabb, abba, abbb, babb, ...
Examples of words that do not belong to language L_5: ε, a, b, aa, ab, ba, bb, aaa, ...

Exercise 2: Let us assume $\Sigma = \{a,b\}$ and $n \in \mathbb{N}$.

a) How many words in Σ^* are of length n?

Solution: 2^n

b) How many words in Σ^* are of length at most n?

Solution:

$$2^0 + 2^1 + \cdots + 2^n = \sum_{i=0}^{n} 2^i = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1$$

Exercise 3: Consider the following languages:

$L_1 = \{w \in \{0,1\}^* \mid \text{in } w \text{ is every 0 (directly) followed by 1}\}$
$L_2 = \{w \in \{0,1\}^* \mid w = w^R\}$

a) Enumerate the first 5 words of each of languages L_1, L_2 (the smallest words with respect to order $<_L$).

Solution:

L_1: ε, 1, 01, 11, 011
L_2: ε, 0, 1, 00, 11

b) Enumerate the first 5 words of each of languages $\overline{L_1}$, $\overline{L_2}$.

Solution:

$\overline{L_1}$: 0, 00, 10, 000, 001
$\overline{L_2}$: 01, 10, 001, 011, 100

c) Enumerate the first 5 words of language $L_1 \cap L_2$.

Solution:

$L_1 \cap L_2$: ε, 1, 11, 101, 111

d) Enumerate the first 5 words of language $L_1 \cup L_2$.

Solution:

$L_1 \cup L_2$: ε, 0, 1, 00, 01

Exercise 4: For each of the following pairs of languages L_1 and L_2 write down all words from the concatenation of these languages, i.e., from the language $L_1 \cdot L_2$:

a) $L_1 = \{\varepsilon, abb, bba\}$, $L_2 = \{a, b, abba\}$

Solution: $L_1 \cdot L_2 = \{a, b, abba, abbb, abbabba, bbaa, bbab, bbaabba\}$

b) $L_1 = \{0, 001, 111\}$, $L_2 = \{\varepsilon, 01, 0101\}$

c) $L_1 = \{aa, aaaa, aaaaa, aaaaaa\}$, $L_2 = \{aa, aaa\}$

d) $L_1 = \emptyset$, $L_2 = \{011, 1111, 010101\}$

e) $L_1 = \{\varepsilon, a, ba, baa\}$, $L_2 = \{\varepsilon\}$

Exercise 5: Consider languages over the alphabet $\{0, 1\}$. Describe the language of all words in the iteration $\{00, 111\}^*$ and write the first 10 words of the language.

Solution: The language contains exactly those words that can be divided into sequences of zeroes of even length and sequences of ones of length divisible by three.

Exercise 6: Consider the following languages:

$$L_1 = \{w \in \{0,1\}^* \mid |w|_1 \leq 1\}$$
$$L_2 = \{w \in \{0,1\}^* \mid w = w^R\}$$

Describe the words in the language $L_1 \cap L_2$.

Solution: Those words with 0s only, or having just one 1 in the middle, i.e., ε, 0, 00, 000, $\ldots$, 1, 010, 00100, $\ldots$

Exercise 7: Determine, which of the following propositions hold generally for all languages. In the case that the given proposition holds for all languages, give a justification why it is the case. In the case that the proposition does not hold generally, give a concrete example of languages for which the given proposition does not hold:

a) $L_1 \cdot L_2 = L_2 \cdot L_1$

   Solution: It does not hold in general.

b) $L_1 \cdot (L_2 \cdot L_3) = (L_1 \cdot L_2) \cdot L_3$

   Solution: It holds.

c) $L_1 \cdot (L_2 \cup L_3) = L_1 \cdot L_2 \, \cup \, L_1 \cdot L_3$

   Solution: It holds.

d) $L_1 \cdot (L_2 \cap L_3) = L_1 \cdot L_2 \, \cap \, L_1 \cdot L_3$

   Solution: It does not hold in general.

e) $(L_1 \cup L_2)^* = L_1^* \cup L_2^*$

   Solution: It does not hold in general.

Exercise 8: Describe at least 5 different orders on the set of all words over the alphabet $\{0,1\}$. For each of them show that it really is an order, i.e., a reflexive, transitive and antisymmetric relation in the case of nonstrict order, or transitive and antisymmetric relation in the case of strict order, and then describe in detail properties of the given order (for example, if it is a total or partial order, if there is some least or greatest element, which elements are minimal or maximal, if there exist infinite decreasing or increasing sequences, etc.)

Solution: For example the following orderings $\sqsubseteq_1$, $\ldots$, $\sqsubseteq_5$:

a) to be a prefix: $x \sqsubseteq_1 y$ iff $\exists z \in \Sigma^* : xz = y$

b) to be a subword: $x \sqsubseteq_2 y$ iff $\exists z_1, z_2 \in \Sigma^* : z_1 x z_2 = y$

c) to be a subsequence: $x \sqsubseteq_3 y$ iff $\exists k \geq 1 : \exists x_1, x_2, \ldots, x_k, u_1, u_2, \ldots, u_{k+1} \in \Sigma^* : x = x_1 x_2 \cdots x_k \, \wedge \, y = u_1 x_1 u_2 x_2 u_3 \cdots u_k x_k u_{k+1}$

d) lexicographic ordering: $x \sqsubseteq_4 y$ iff $x \sqsubseteq_1 y$ or $\exists a, b \in \Sigma : \exists u, v, w \in \Sigma^* : x = uav \wedge y = ubw \wedge a < b$ (we consider some fixed ordering $<$ on symbols of alphabet Σ; in particular, for $\Sigma = \{0, 1\}$ we can take $0 < 1$)

e) ordering by the length of words, and for words of the same length, lexicographic ordering (it was defined in a lecture as $<_L$): $x \sqsubseteq_5 y$ iff $|x| < |y| \vee (|x| = |y| \wedge x \sqsubseteq_4 y)$