NP-Complete Problems

Polynomial Algorithms

An algorithm is **polynomial** if its time complexity is polynomial, i.e., if it is in $O(n^k)$ where k is a constant.

The notion of “polynomial algorithm” can be viewed as a certain approximation of what algorithms are generally viewed as “efficient” and useable in practive for quite long inputs.

Remark: Algorithms that are not polynomial (i.e., that have a greater time complexity than polynomial, e.g., exponential) are generally not viewed as efficient.

Such algorithms with nonpolynomial time complexity can be usually used only for “small” inputs.

Polynomial Algorithms

However, we must be aware of the following:

- An algorithm whose time complexity is for example in $\Theta(n^{100})$ surely can not be viewed as effiecient from a practical point of view.
- It can be shown that for each k it is possible to construct an artificial example of an algorithmic problem that can be solved using an algorithm with time complexity in $O(n^{k+1})$ but there with no algorithm with time complexity in $O(n^k)$.
- For "naturally" defined problems that are solved in practice it is not the case that for there would be some polynomial algorithm with a big degree of a polynomial and would not be some algorithm with a small degree of polynomial.

  Usually, we have one of two possibilities:
  - A polynomial algorithm is known and the degree of the polynomial is quite small, e.g., at most 5.
  - There is no known algorithm for the given problem.

Polynomial Algorithms

- From the practical point of view, sometimes even an algorithm with time complexity for example $\Theta(n^2)$ can be viewed as inefficient for some purpose — e.g., for extremely bigs inputs or if there are some very strict timing constraints.
- On the other hand, for some purposes even an algorithm with exponential time complexity can sometimes useful in practice.
- There are examples of algorithms that have an an exponential time complexity in the worst case but for many inputs they actually work efficiently and they can be used to process quite big inputs.

Class PTIME

Complexity classes — sets of those algorithmic problems, for which there exist algorithms with a certain computational complexity.

The class **PTIME** is the class of those algorithmic decision problems, for which there exists an algorithm with a polynomial time complexity.

Class PTIME

Note that the definition of the class PTIME is robust in the sense that which problems belong to this class does not depend much on what model of computation we consider.

For all “reasonable” sequential models of computation, this class contains the same problems.

Remark: As “resonable” sequential models of computation are considered those that can be simulated by Turing machines in such a way that the running time increases only polynomially in such simulation.

“Reasonable” Sequential Models of Computation

Examples of models of computation considered to be “reasonable” from this point of view:

- variants of Turing machines (one-tape, multi-tape, . . .)
- RAMs with the use of logarithmic measure
- RAMs without operations for multiplication and division with the use of unit measure
- RAMs that have operations for multiplication and division with the use unit measure if it is ensured for the given RAM that during a computation each memory cell contains a number whose size is bounded by some polynomial

Models of computation that are not “reasonable”

Examples of models of computation that are not “reasonable” from this point of view:

- RAMs with operations form multiplication and division using the unit measure (without restrictions on the size of numbers, on which arithmetic operations can be performed in one step) — they can perform in one step perform arithmetic operations on numbers that have exponential number of bits
- Minsky machines — they are too slow, execution of simple operations takes too much time; in a simulation of a computation of a Turing machine, the time grows exponentially with respect to the original Turing machine

Nondeterministic Algorithms

So far, we have considered only deterministic algorithms.

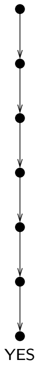
We can also consider **nondeterministic** algorithms performed by nondeterministic variants of various kinds of machines:

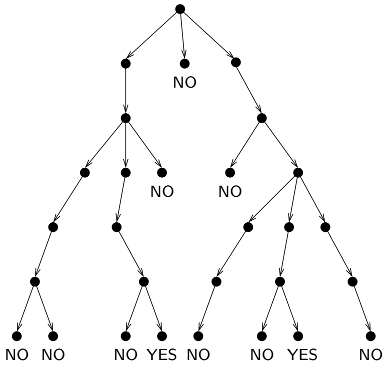
- Turing machines
- RAMs
- ...

The running time for nondeterministic variants of machines is specified as the the running time of the longest possible computation of a given machine for a given input.

Nondeterministic Algorithms

A nondeterministic algorithm gives the answer YES for a given input x if there **exists** at least one computation of this machine that gives answer YES.





Nondeterministic Algorithms

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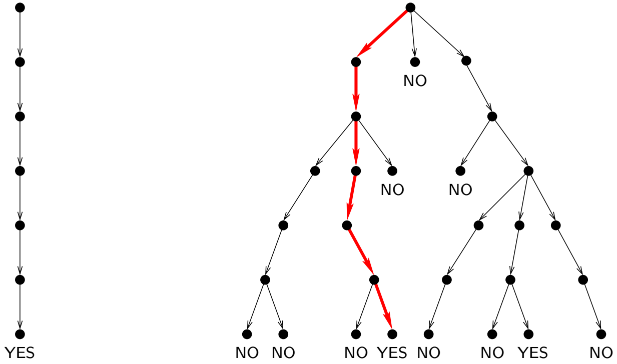


Problem SAT

SAT (boolean satisfiability problem)

Input: Boolean formula φ.

Question: Is φ satisfiable?

Example:
Formula $\varphi_1 = x_1 \wedge (\neg x_2 \vee x_3)$ is satisfiable:
e.g., for valuation v where $v(x_1) = 1$, $v(x_2) = 0$, $v(x_3) = 1$, the formula φ_1 is true.

Formula $\varphi_2 = (x_1 \wedge \neg x_1) \vee (\neg x_2 \wedge x_3 \wedge x_2)$ is not satisfiable:
it is false for every valuation v.

Nondeterministic Algorithms

A nondeterministic algorithm solving the SAT problem in polynomial time:

- It reads a formula φ.
- It cycles through all variables $x_1, x_2, \ldots, x_k$occurring in the formula φ.
  For each of these variables, it nondeterministically chooses if it assigns value 0 or value 1 to it.
- It evaluates the truth value of formula φ for the generated valuation.
  It halts with answer $\textsc{Yes}$ if the formula φ is true for the give valuation.
  Otherwise, it halts with answer $\textsc{No}$.

Nondeterministic Algorithms

A behaviour of a nondeterministic algorithm can be easily simulated by a deterministic algorithm:

- the deterministic algorithm systematically simulates the behaviour of all individual branches of the nondeterministic computation — it traverses the tree of all computations in a depth-first manner
- in this traversal, it uses a stack where it stores for each configuration on the currently searched branch information that will allow it to return to the previous configuration from the current configuration

Using this simple simulation, the running time of the deterministic algorithm grows exponentially with respect to the original nondeterministic algorithm.

Nondeterministic Algorithms

Nondeterministic algorithms can be alternatively viewed as deterministic algorithms that do not solve a given problem but instead they just allow to verify, using some provided additional information they would be given to them, whether the answer for the given input is $\textsc{Yes}$:

- The algorithm expects as an input not just an instance $x \in \mathit{In}$ of the given problem but also an additional information y — a **witness** that the answer is $\textsc{Yes}$.
- A deterministic algorithm performs a computation for the pair (x, y) and returns for it an answer $\textsc{Yes}$ or $\textsc{No}$.
- The following must hold for each instance $x \in \mathit{In}$:
  - If the correct answer for the instance x is $\textsc{Yes}$ then there exists at least one witness y such that the algorithm returns answer $\textsc{Yes}$ for the pair (x, y).
  - If the correct answer for the instance x is $\textsc{No}$ then for each potential witness y, the algorithm return answer $\textsc{No}$ for the pair (x, y).

Nondeterministic Algorithms

In the case of nondeterministic polynomial algorithms, the following holds:

- The size of each potential witness is polynomial with respect to the size of the original instance x.

  I.e., there is a polynomial p such that for each instance x and each potential witness y it holds that $|y| \leq p(|x|)$.

- The time complexity of deterministic algorithm verifying a given pair (x, y) is polynomial.

Example: For the SAT problem, a witness proving that the answer is $\textsc{Yes}$ (i.e., that the given formula is satisfiable) a truth valuation, for which the formula is true.

Class NPTIME

The class **NPTIME** is the class of all algorithmic decision problems, for which there exists a **nondeterministic** algorithm with a polynomial time complexity.

Example: It is obvious that the SAT problem belongs to the class NPTIME.

Does SAT also belong to the class PTIME ?

Remark: The fact that each problem that belongs to PTIME belongs also to NPTIME, is obvious.

NP-Complete Problems

There is a big class of algorithmic problems called **NP-complete** problems such that:

- they can be solved by a nondeterministic algorithm with a polynomial time complexity (and so also by a deterministic algorithm with an exponential time complexity)
- no polynomial time algorithm is known for any of these problems
- on the other hand, for any of these problems it is not proved that there cannot exist a polynomial time algorithm for the given problem
- every NP-complete problem can be polynomially reduced to any other NP-complete problem

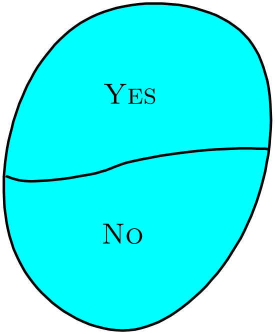
Remark: This is not a definition of NP-complete problems. The precise definition will be described later.

Polynomial Reductions between Problems

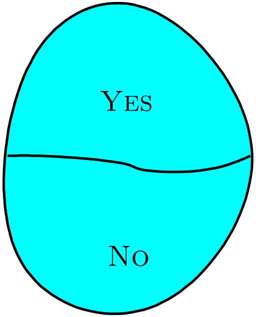
There is a **polynomial reduction** of problem P_1 to problem P_2 if there exists an algorithm Alg with a polynomial time complexity that reduces problem P_1 to problem P_2.

Polynomial Reductions between Problems

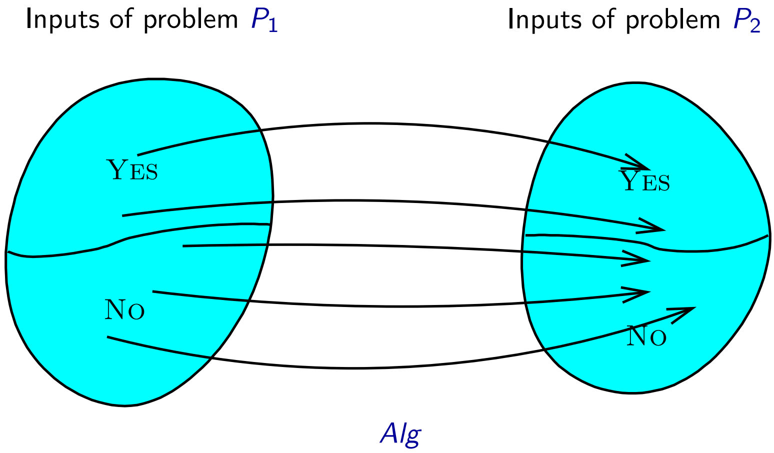
Inputs of problem P_1



Inputs of problem P_2



Polynomial Reductions between Problems



Polynomial Reductions between Problems

Let us say that problem A can be reduced in polynomial time to problem B, i.e., there is a polynomial algorithm P realizing this reduction.

If problem B is in the class PTIME then problem A is also in the class PTIME.

A solution of problem A for an input x:

- Call P with input x and obtain a returned value $P(x)$.
- Call a polynomial time algorithm solving problem B with the input $P(x)$.
  Write the returned value as the answer for A.

That means:

If A is not in PTIME then also B can not be in PTIME.

Polynomial Reductions between Problems

As an example, a polynomial time reduction from the 3-SAT problem to the independent set problem (IS) will be described.

Remark: Both 3-SAT and IS are examples of NP-complete problems.

Problem 3-SAT

3-SAT is a variant of the SAT problem where the possible inputs are restricted to formulas of a certain special form:

3-SAT

Input: Formula φ is a conjunctive normal form where every clause contains exactly 3 literals.

Question: Is φ satisfiable?

Problem 3-SAT

Recalling some notions:

- A **literal** is a formula of the form x or $\neg x$ where x is an atomic proposition.
- A **clause** is a disjuction of literals.

  Examples: $x_1 \vee \neg x_2 \qquad \neg x_5 \vee x_8 \vee \neg x_{15} \vee \neg x_{23} \qquad x_6$
- A formula is in a **conjuctive normal form (CNF)** if it is a conjuction of clauses.

  Example: $(x_1 \vee \neg x_2) \ \wedge \ (\neg x_5 \vee x_8 \vee \neg x_{15} \vee \neg x_{23}) \ \wedge \ x_6$

So in the 3-SAT problem we require that a formula φ is in a CNF and moreover that every clause of φ contains exactly three literals.

Example:

$(x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee x_3) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Problem 3-SAT

The following formula is satisfiable:

$(x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_3 \vee x_3) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_4)$

It is true for example for valuation v where

$$\begin{aligned} v(x_1) &= 0 \\ v(x_2) &= 1 \\ v(x_3) &= 0 \\ v(x_4) &= 1 \end{aligned}$$

On the other hand, the following formula is not satisfiable:

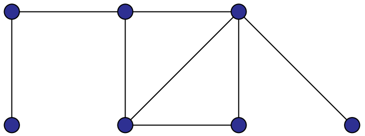
$$(x_1 \vee x_1 \vee x_1) \wedge (\neg x_1 \vee \neg x_1 \vee \neg x_1)$$

Independent Set (IS) Problem

Independent set (IS) problem

Input: An undirected graph G, a number k.

Question: Is there an independent set of size k in the graph G?



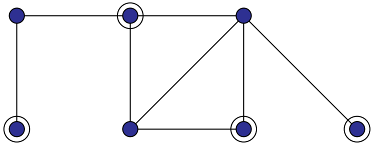
$k = 4$

Remark: An **independent set** in a graph is a subset of nodes of the graph such that no pair of nodes from this set is connected by an edge.

Independent Set (IS) Problem

Independent set (IS) problem

Input:	An undirected graph G, a number k.
Question:	Is there an independent set of size k in the graph G?

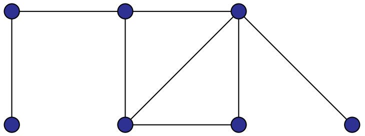


$k = 4$

Remark: An **independent set** in a graph is a subset of nodes of the graph such that no pair of nodes from this set is connected by an edge.

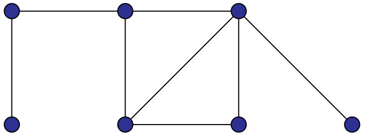
Independent Set (IS) Problem

An example of an instance where the answer is YES:



$k = 4$

An example of an instance where the answer is NO:



$k = 5$

A Reduction from 3-SAT to IS

We describe a (polynomial-time) algorithm with the following properties:

- **Input:** An arbitrary instance of 3-SAT, i.e., a formula φ in a conjunctive normal form where every clause contains exactly three literals.
- **Output:** An instance of IS, i.e., an undirected graph G and a number k.
- Moreover, the following will be ensured for an arbitrary input (i.e., for an arbitrary formula φ in the above mentioned form):

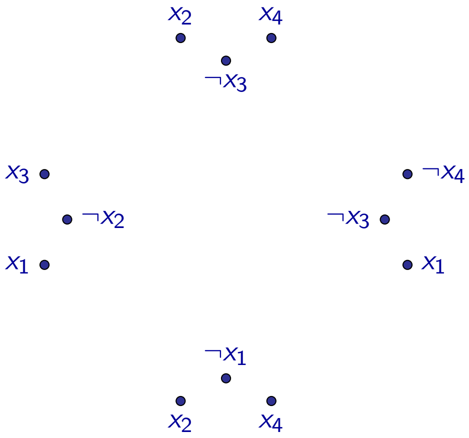
  There will be an independent set of size k in graph G iff formula φ will be satisfiable.

A Reduction from 3-SAT to IS

$$(x_1 \vee \neg x_2 \vee x_3) \;\wedge\; (x_2 \vee \neg x_3 \vee x_4) \wedge\; (x_1 \vee \neg x_3 \vee \neg x_4) \wedge\; (\neg x_1 \vee x_2 \vee x_4)$$

A Reduction from 3-SAT to IS

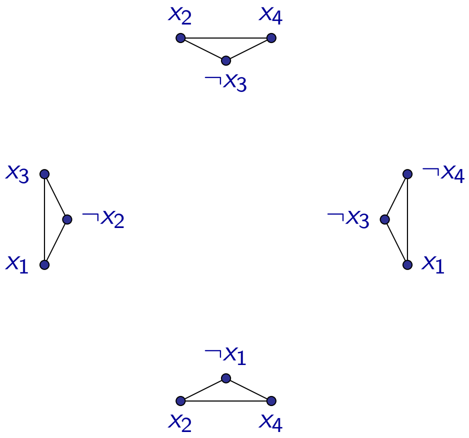
$(x_1 \vee \neg x_2 \vee x_3) \ \wedge \ (x_2 \vee \neg x_3 \vee x_4) \wedge \ (x_1 \vee \neg x_3 \vee \neg x_4) \wedge \ (\neg x_1 \vee x_2 \vee x_4)$



For each occurrence of a literal we add a node to the graph.

A Reduction from 3-SAT to IS

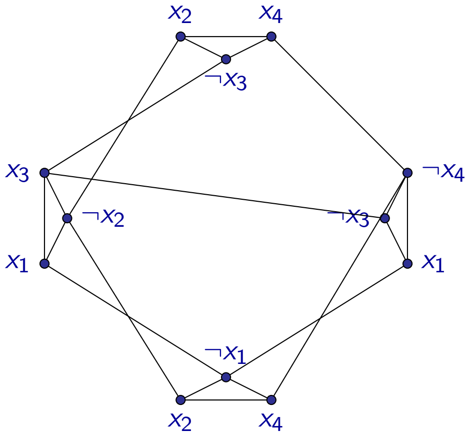
$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_2 \vee x_4)$



We connect with edges the nodes corresponding to occurrences of literals belonging to the same clause.

A Reduction from 3-SAT to IS

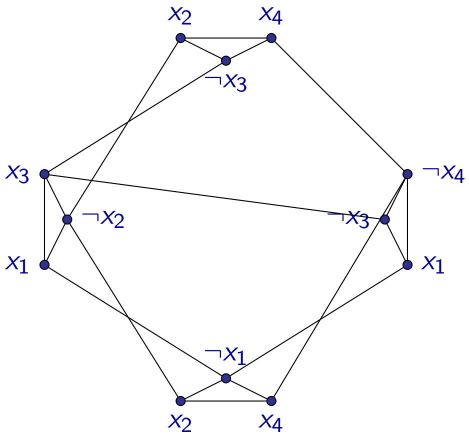
$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_2 \vee x_4)$



For each pair of nodes corresponding to literals x_i and $\neg x_i$ we add an edge between them.

A Reduction from 3-SAT to IS

$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_2 \vee x_4)$

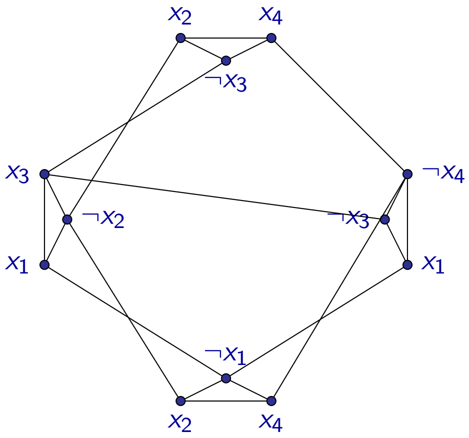


$k = 4$

We put k to be equal to the number of clauses.

A Reduction from 3-SAT to IS

$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_2 \vee x_4)$



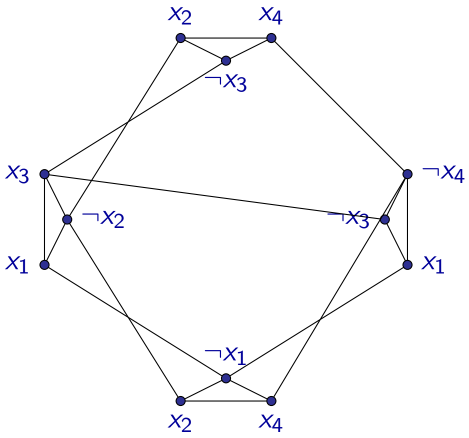
$k = 4$

The constructed graph and number k are the output of the algorithm.

A Reduction from 3-SAT to IS

$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_2 \vee x_4)$

$v(x_1) = 1$
$v(x_2) = 1$
$v(x_3) = 0$
$v(x_4) = 1$



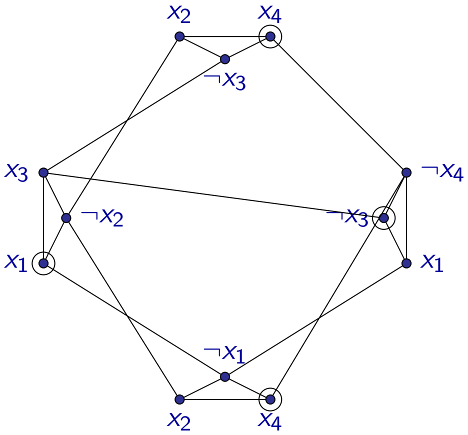
$k = 4$

If the formula φ is satisfiable then there is a valuation v where every clause contains at least one literal with value 1.

A Reduction from 3-SAT to IS

$(x_1 \vee \neg x_2 \vee x_3) \ \wedge\ (x_2 \vee \neg x_3 \vee x_4) \ \wedge\ (x_1 \vee \neg x_3 \vee \neg x_4) \ \wedge\ (\neg x_1 \vee x_2 \vee x_4)$

$v(x_1) = 1$
$v(x_2) = 1$
$v(x_3) = 0$
$v(x_4) = 1$



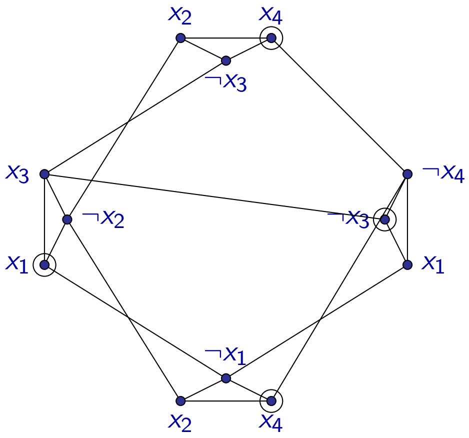
$k = 4$

We select one literal that has a value 1 in the valuation v, and we put the corresponding node into the independent set.

A Reduction from 3-SAT to IS

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4) \wedge (x_1 \vee \neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_2 \vee x_4)$$

$v(x_1) = 1$
$v(x_2) = 1$
$v(x_3) = 0$
$v(x_4) = 1$



$k = 4$

We can easily verify that the selected nodes form an independent set.

A Reduction from 3-SAT to IS

The selected nodes form an independent set because:

- One node has been selected from each triple of nodes corresponding to one clause.
- Nodes denoted x_i and $\neg x_i$ could not be selected together.
  (Exactly of them has the value 1 in the given valuation v.)

A Reduction from 3-SAT to IS

On the other hand, if there is an independent set of size k in graph G, then it surely has the following properties:

- At most one node is selected from each triple of nodes corresponding to one clause.

  But because there are k clauses and k nodes are selected, exactly one node must be selected from each triple.

- Nodes denoted x_i and $\neg x_i$ cannot be selected together.

We can choose a valuation according to the selected nodes, since it follows from the previous discussion that it must exist.
(Arbitrary values can be assigned to the remaining variables.)

For the given valuation, the formula φ has surely the value 1, since in each clause there is at least one literal with value 1.

A Reduction from 3-SAT to IS

It is obvious that the running time of the described algorithm polynomial:

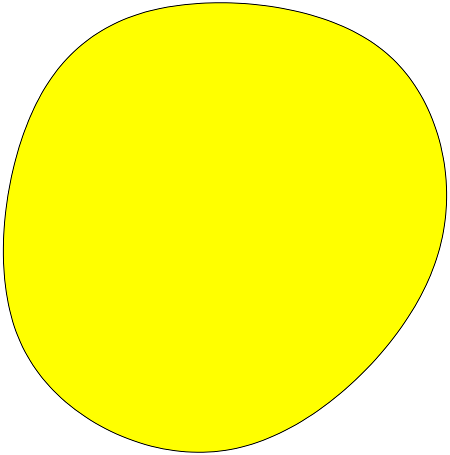
Graph G and number k can be constructed for a formula φ in time $O(n^2)$, where n is the size of formula φ.

We have also seen that there is an independent set of size k in the constructed graph G iff the formula φ is satisfiable.

The described algorithm shows that 3-SAT can be reduced in polynomial time to IS.

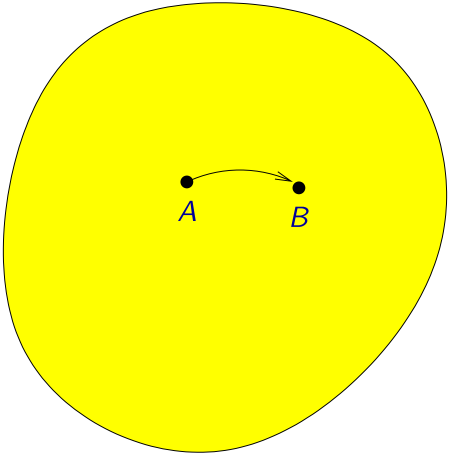
NP-Complete Problems

Let us consider a set of all decision problems.



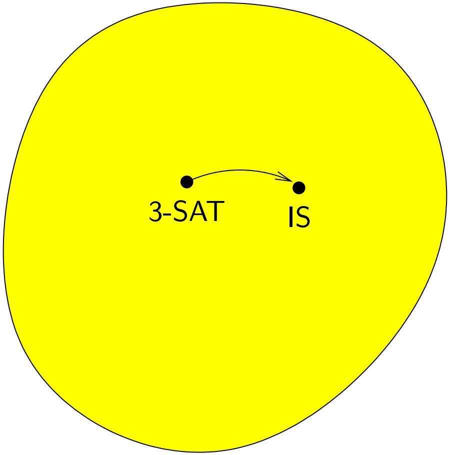
NP-Complete Problems

By an arrow we denote that a problem A can be reduced in polynomial time to a problem B.



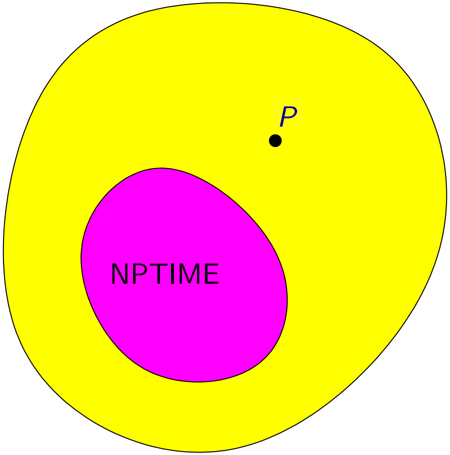
NP-Complete Problems

For example 3-SAT can be reduced in polynomial time to IS.



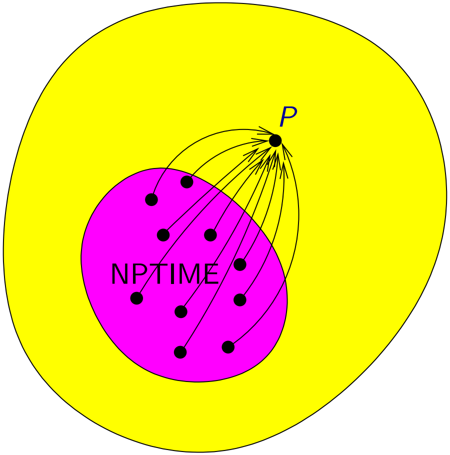
NP-Complete Problems

Let us consider now the class NPTIME and a problem P.



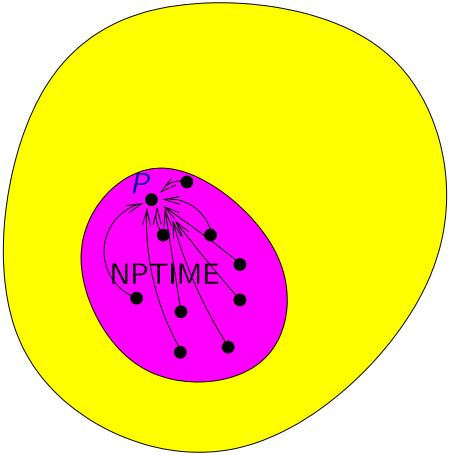
NP-Complete Problems

A problem P is NP-**hard** if every problem from NPTIME can be reduced in polynomial time to P.



NP-Complete Problems

A problem P is **NP-complete** if it is NP-hard and it belongs to the class NPTIME.



NP-Complete Problems

If we have found a polynomial time algorithm for some NP-hard problem P, then we would have polynomial time algorithms for all problems P' from NPTIME:

- At first we would apply an algorithm for the reduction from P' to P on an input of a problem P'.
- Then we would use a polynomial algorithm for P on the constructed instance of P and returned its result as the answer for the original instance of P'.

Is such case, PTIME $=$ NPTIME would hold, since for every problem from NPTIME there would be a polynomial-time (deterministic) algorithm.

NP-Complete Problems

On the other hand, if there is at least one problem from NPTIME for which a polynomial-time algorithm does not exist, then it means that for none of NP-hard problems there is a polynomial-time algorithm.

It is an open question whether the first or the second possibility holds.

NP-Complete Problems

It is not difficult to see that:

If a problem A can be reduced in a polynomial time to a problem B and problem B can be reduced in a polynomial time to a problem C, then problem A can be reduced in a polynomial time to problem C.

So if we know about some problem P that it is NP-hard and that P can be reduced in a polynomial time to a problem P', then we know that the problem P' is also NP-hard.

NP-Complete Problems

Theorem (Cook, 1971)

Problem SAT is NP-complete.

It can be shown that SAT can be reduced in a polynomial time to 3-SAT and we have seen that 3-SAT can be reduced in a polynomial time to IS.

This means that problems 3-SAT and IS are NP-hard.

It is not difficult to show that 3-SAT and IS belong to the class NPTIME.

Problems 3-SAT and IS are NP-complete.

NP-Complete Problems

By a polynomial reductions from problems that are already known to be NP-complete, NP-completeness of many other problems can be shown:

$$\begin{tikzcd}
 & & & \text{VC} & & \\
 & & & \text{CLIQUE} & & \\
 & & \text{IS} \arrow[uur] \arrow[ur] \arrow[dr] & & & \\
\text{SAT} \arrow[r] & \text{3–SAT} \arrow[ur] \arrow[dr] \arrow[ddr] \arrow[dddr] & & \text{HC} \arrow[r] & \text{HK} \arrow[r] & \text{TSP} \\
 & & \text{3–CG} & & & \\
 & & \text{SUBSET–SUM} & & & \\
 & & \text{ILP} & & &
\end{tikzcd}$$

Examples of Some NP-Complete Problems

The following previously mentioned problems are NP-complete:

- SAT (boolean satisfiability problem)
- 3-SAT
- IS — independent set problem

On the following slides, examples of some other NP-complete problems are described:

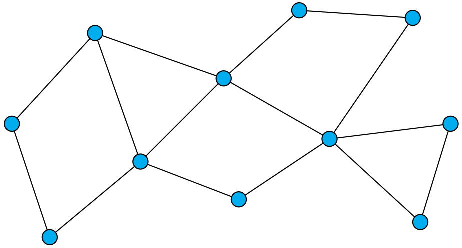
- CG — graph coloring (remark: it is NP-complete even in the special case where we have 3 colors)
- VC — vertex cover
- CLIQUE — clique problem
- HC — Hamiltonian cycle
- HK — Hamiltonian circuit
- TSP — traveling salesman problem
- SUBSET-SUM
- ILP — integer linear programming

Graph Coloring

Graph coloring

Input:	An undirected graph G, a natural number k.
Question:	Is it possible to color nodes of the graph G using k colors in such a way that there is no pair of nodes where both nodes are colored with the same color and connected with an edge?

Example: $k = 3$



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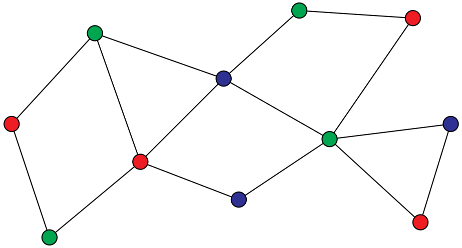

Answer: YES

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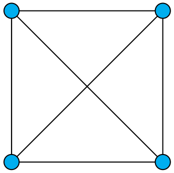

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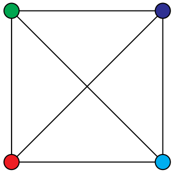


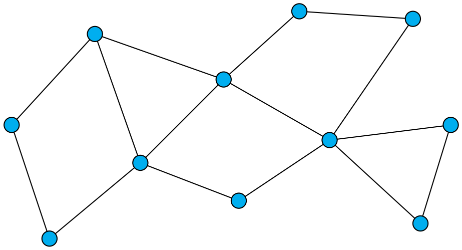
Answer: No

VC – Vertex Cover

VC – vertex cover

Input:	An undirected graph G and a natural number k.
Question:	Is there some subset of nodes of G of size k such that every edge has at least one of its nodes in this subset?

Example: $k = 6$



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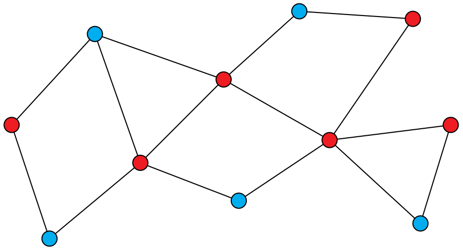


Answer: YES

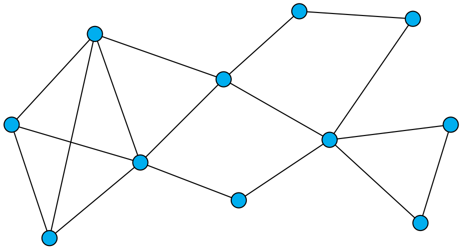
CLIQUE

CLIQUE

Input: An undirected graph G and a natural number k.

Question: Is there some subset of nodes of G of size k such that every two nodes from this subset are connected by an edge?

Example: $k = 4$



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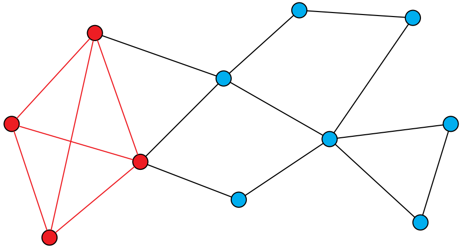


Answer: YES

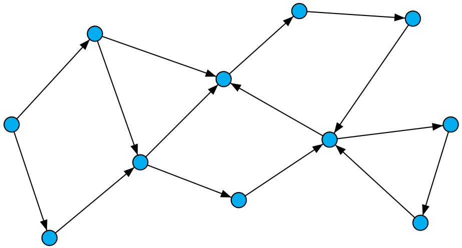
Hamiltonian Cycle

HC – Hamiltonian cycle

Input: A directed graph G.

Question: Is there a Hamiltonian cycle in G (i.e., a directed cycle going through each node exactly once)?

Example:



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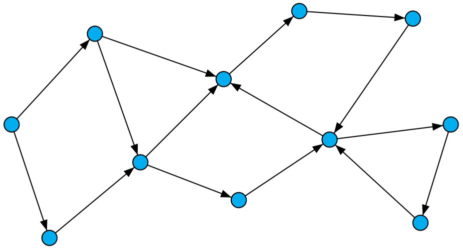

Answer: NO

Hamiltonian Cycle

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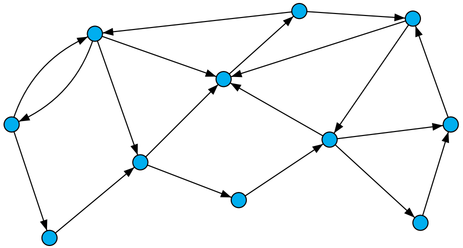

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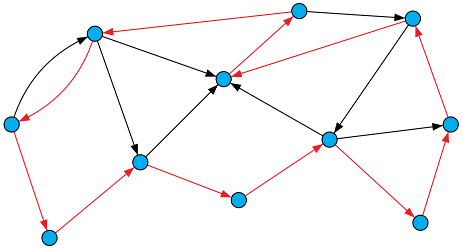


Answer: YES

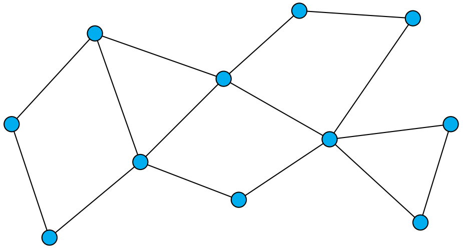
Hamiltonian Circuit

HK – Hamiltonian circuit

Input: An undirected graph G.

Question: Is there a Hamiltonian circuit in G (i.e., an undirected cycle going through each node exactly once)?

Example:



Answer: No

Hamiltonian Circuit

HK – Hamiltonian circuit

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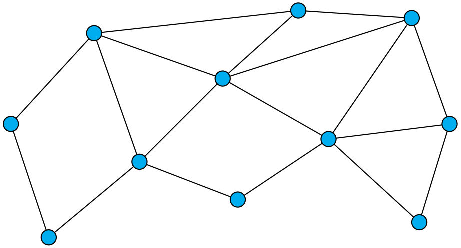

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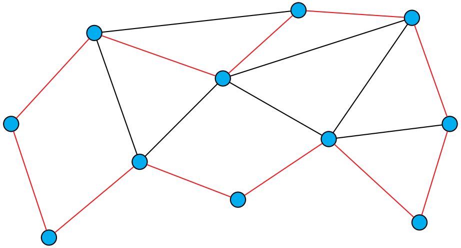


Answer: YES

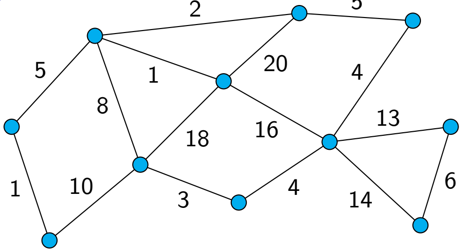
Traveling Salesman Problem

TSP - traveling salesman problem

Input: An undirected graph G with edges labelled with natural numbers and a number k.

Question: Is there a closed tour going through all nodes of the graph G such that the sum of labels of edges on this tour is at most k?

Example: $k = 70$



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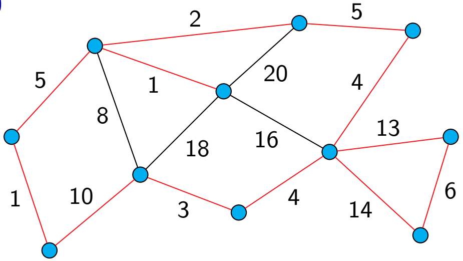

Answer: YES, since there is a tour with the sum 69.

SUBSET-SUM

Problem SUBSET-SUM

Input: A sequence $a_1, a_2, \ldots, a_n$ of natural numbers and a natural number s.

Question: Is there a set $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} a_i = s$?

In other words, the question is whether it is possible to select a subset with sum s of a given (multi)set of numbers.

Example: For the input consisting of numbers $3, 5, 2, 3, 7$ and number $s = 15$ the answer is YES, since $3 + 5 + 7 = 15$.

For the input consisting of numbers $3, 5, 2, 3, 7$ and number $s = 16$ the answer is NO, since no subset of these numbers has sum 16.

SUBSET-SUM

Remark:
The order of numbers $a_1, a_2, \ldots, a_n$ in an input is not important.

Note that this is not exactly the same as if we have formulated the problem so that the input is a set $\{a_1, a_2, \ldots, a_n\}$ and a number s — numbers cannot occur multiple times in a set but they can in a sequence.

SUBSET-SUM

Problem SUBSET-SUM is a special case of a **knapsack problem**:

Knapsack problem

Input: Sequence of pairs of natural numbers $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ and two natural numbers s and t.

Question: Is there a set $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} a_i \leq s$ and $\sum_{i \in I} b_i \geq t$?

SUBSET-SUM

Informally, the knapsack problem can be formulated as follows:

We have n objects, where the i-th object weights a_i grams and its price is b_i dollars.

The question is whether there is a subset of these objects with total weight at most s grams (s is the capacity of the knapsack) and with total price at least t dollars.

Remark:

Here we have formulated this problem as a decision problem.

This problem is usually formulated as an optimization problem where the aim is to find such a set $I \subseteq \{1, 2, \ldots, n\}$, where the value $\sum_{i \in I} b_i$ is maximal and where the condition $\sum_{i \in I} a_i \leq s$ is satisfied, i.e., where the capacity of the knapsack is not exceeded.

SUBSET-SUM

That SUBSET-SUM is a special case of the Knapsack problem can be seen from the following simple construction:

Let us say that $a_1, a_2, \ldots, a_n$, s_1 is an instance of SUBSET-SUM.
It is obvious that for the instance of the knapsack problem where we have the sequence (a_1, a_1), $(a_2, a_2), \ldots,$ (a_n, a_n), $s = s_1$ and $t = s_1$, the answer is the same as for the original instance of SUBSET-SUM.

SUBSET-SUM

If we want to study the complexity of problems such as SUBSET-SUM or the knapsack problem, we must clarify what we consider as the size of an instance.

Probably the most natural it is to define the size of an instance as the total number of bits needed for its representation.

We must specify how natural numbers in the input are represented – if in binary (resp. in some other numeral system with a base at least 2 (e.g., decimal or hexadecimal) or in unary.

- If we consider the total number of bits when numbers are written in **binary** as the size of an input, no polynomial time algorithm is known for SUBSET-SUM.
- If we consider the total number of bits when numbers are written in **unary** as the size of an input, SUBSET-SUM can be solved by an algorithm whose time complexity is polynomial.

ILP – Integer Linear Programming

Problem ILP (integer linear programming)

Input:	An integer matrix A and an integer vector b.
Question:	Is there an integer vector x such that $Ax \leq b$?

An example of an instance of the problem:

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix}$$

So the question is if the following system of inequations has some integer solution:

$$\begin{aligned} 3x_1 - 2x_2 + 5x_3 &\leq 8 \\ x_1 + x_3 &\leq -3 \\ 2x_1 + x_2 &\leq 5 \end{aligned}$$

ILP – Integer Linear Programming

One of solutions of the system

$$\begin{array}{rcl} 3x_1 - 2x_2 + 5x_3 & \leq & 8 \\ x_1 + x_3 & \leq & -3 \\ 2x_1 + x_2 & \leq & 5 \end{array}$$

is for example $x_1 = -4$, $x_2 = 1$, $x_3 = 1$, i.e.,

$$x = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$$

because

$$\begin{array}{rcrcr} 3 \cdot (-4) - 2 \cdot 1 + 5 \cdot 1 & = & -9 & \leq & 8 \\ -4 + 1 & = & -3 & \leq & -3 \\ 2 \cdot (-4) + 1 & = & -7 & \leq & 5 \end{array}$$

So the answer for this instance is YES.

ILP – Integer Linear Programming

Remark: A similar problem where the question for a given system of linear inequations is whether it has a solution in the set of **real** numbers, can be solved in a polynomial time.