Regular Expressions

Regular Expressions

Regular expressions describing languages over an alphabet Σ:

- $\emptyset$, ε, a (where $a \in \Sigma$) are regular expressions:

  $\emptyset$ ... denotes the empty language
  ε ... denotes the language $\{\varepsilon\}$
  a ... denotes the language $\{a\}$

- If α, β are regular expressions then also $(\alpha + \beta)$, $(\alpha \cdot \beta)$, (α^*) are regular expressions:

  $(\alpha + \beta)$ ... denotes the union of languages denoted α and β
  $(\alpha \cdot \beta)$ ... denotes the concatenation of languages denoted α and β
  (α^*) ... denotes the iteration of a language denoted α

- There are no other regular expressions except those defined in the two points mentioned above.

Regular Expressions

Example: alphabet $\Sigma = \{0, 1\}$

- According to the definition, $\mathtt{0}$ and $\mathtt{1}$ are regular expressions.

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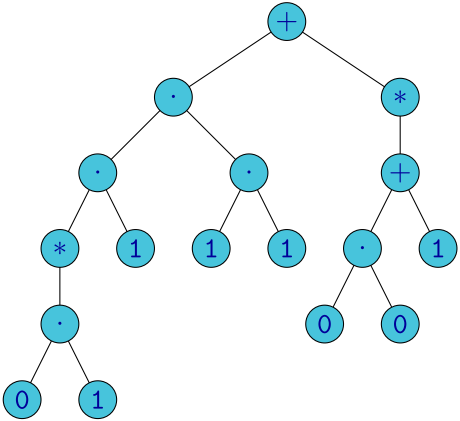
Remark: If α is a regular expression, by $\mathcal{L}(\alpha)$ we denote the language defined by the regular expression α.

$$\mathcal{L}(((0 + 1) \cdot (0^*))) = \{0,\ 1,\ 00,\ 10,\ 000,\ 100,\ 0000,\ 1000,\ 00000,\ \ldots\}$$

Regular Expressions

The structure of a regular expression can be represented by an abstract syntax tree:



$$(((((0 \cdot 1)^*) \cdot 1) \cdot (1 \cdot 1)) + (((0 \cdot 0) + 1)^*))$$

Regular Expressions

The formal definition of semantics of regular expressions:

- $\mathcal{L}(\emptyset) = \emptyset$
- $\mathcal{L}(\varepsilon) = \{\varepsilon\}$
- $\mathcal{L}(a) = \{a\}$
- $\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$
- $\mathcal{L}(\alpha \cdot \beta) = \mathcal{L}(\alpha) \cdot \mathcal{L}(\beta)$
- $\mathcal{L}(\alpha + \beta) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta)$

Regular Expressions

To make regular expressions more lucid and succinct, we use the following conventions:

- The outward pair of parentheses can be omitted.
- We can omit parentheses that are superflous due to associativity of operations of union ($+$) and concatenation ($\cdot$).
- We can omit parentheses that are superflous due to the defined priority of operators (iteration (*) has the highest priority, concatenation ($\cdot$) has lower priority, and union ($+$) has the lowest priority).
- A dot denoting concatenation can be omitted.

Example: Instead of

$$(((((0 \cdot 1)^*) \cdot 1) \cdot (1 \cdot 1)) + (((0 \cdot 0) + 1)^*))$$

we usually write

$$(01)^*111 + (00 + 1)^*$$

Regular Expressions

Examples: In all examples $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

$\mathtt{a}$ ... the language containing the only word $\mathtt{a}$

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a ... the language containing the only word a

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$\mathtt{a}^*$ ... the language containing words ε, $\mathtt{a}$, $\mathtt{aa}$, $\mathtt{aaa}$, ...

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$(\mathtt{ab})^*$ ... the language containing words ε, $\mathtt{ab}$, $\mathtt{abab}$, $\mathtt{ababab}$, ...

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Examples: In all examples $\Sigma = \{a, b\}$.

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a^* ... the language containing words ε, a, aa, aaa, ...

$(ab)^*$ ... the language containing words ε, ab, $abab$, $ababab$, ...

$(a + b)^*$ ... the language containing all words over the alphabet $\{a, b\}$

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$(a + b)^*aa$ ... the language containing all words ending with aa

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$(ab)^*bbb(ab)^*$ ... the language containing all words that contain a subword bbb preceded and followed by an arbitrary number of copies of the word ab

Regular Expressions

$(a + b)^*aa + (ab)^*bbb(ab)^*$ ... the language containing all words that either end with aa or contain a subwords bbb preceded and followed with some arbitrary number of words ab

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$(a + b)^* b(a + b)^*$ ... the language of all words that contain at least one occurrence of symbol b

Regular Expressions

$(\mathtt{a} + \mathtt{b})^*\mathtt{aa} + (\mathtt{ab})^*\mathtt{bbb}(\mathtt{ab})^*$ ... the language containing all words that either end with aa or contain a subwords bbb preceded and followed with some arbitrary number of words ab

$(\mathtt{a} + \mathtt{b})^*\mathtt{b}(\mathtt{a} + \mathtt{b})^*$ ... the language of all words that contain at least one occurrence of symbol b

$\mathtt{a}^*(\mathtt{ba}^*\mathtt{ba}^*)^*$ ... the language containg all words with an even number of occurrences of symbol b

Transformation of a Regular Expression to a Finite Automaton

Proposition

Every language that can be represented by a regular expression is regular (i.e., it is accepted by some finite automaton).

Proof: It is sufficient to show how to construct for a given regular expression α a finite automaton accepting the language $\mathcal{L}(\alpha)$.

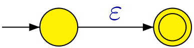
The construction is recursive and proceeds by the structure of the expression α:

- If α is a elementary expression (i.e., $\emptyset$, ε or a):
  - We construct the corresponding automaton directly.
- If α is of the form $(\beta + \gamma)$, $(\beta \cdot \gamma)$ or (β^*):
  - We construct automata accepting languages $\mathcal{L}(\beta)$ and $\mathcal{L}(\gamma)$ recursively.
  - Using these two automata, we construct the automaton accepting the language $\mathcal{L}(\alpha)$.

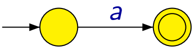
Transformation of a Regular Expression to a Finite Automaton

The automata for the elementary expressions:

$\emptyset$



ε



a

Transformation of a Regular Expression to a Finite Automaton

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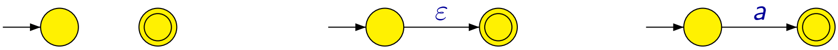
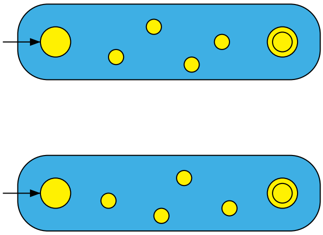


$\emptyset$ ε a

The construction for the union:



Transformation of a Regular Expression to a Finite Automaton

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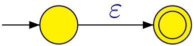
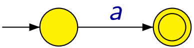


ε



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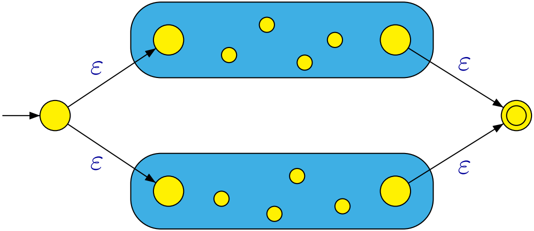
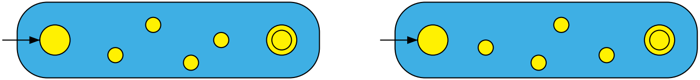


Transformation of a Regular Expression to a Finite Automaton

The construction for the concatenation:



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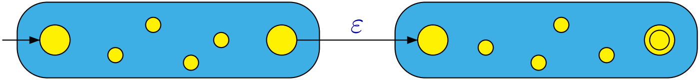

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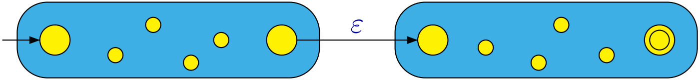
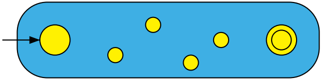


The construction for the iteration:



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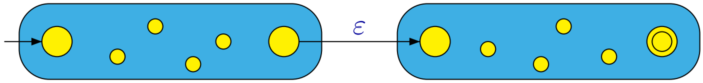

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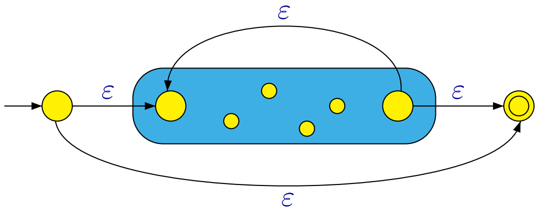

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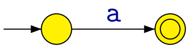

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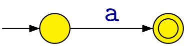

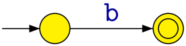

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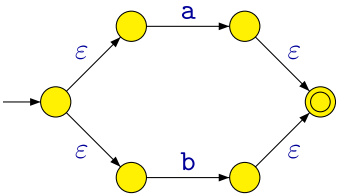

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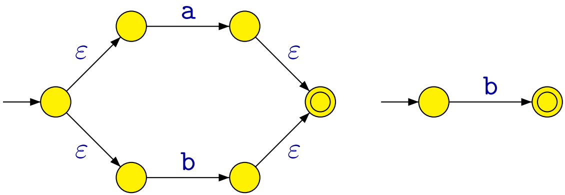

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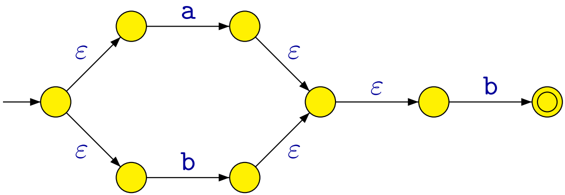

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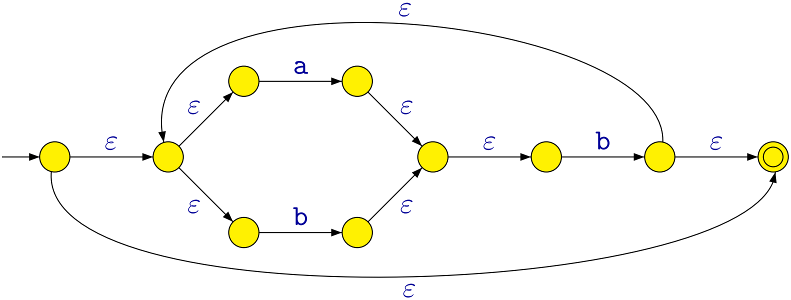


Transformation of a Regular Expression to a Finite Automaton

If an expression α consists of n symbols (not counting parenthesis) then the resulting automaton has:

- at most $2n$ states,
- at most $4n$ transitions.

Remark: By transforming the generalized nondeterministic automaton into a deterministic one, the number of states can grow exponentially, i.e., the resulting automaton can have up to $2^{2n} = 4^n$ states.

Transformation of an Automaton to a Regular Expression

Proposition

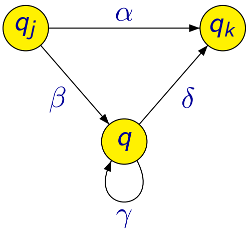
Every regular language can be represented by some regular expression.

Proof: It is sufficient to show how to construct for a given finite automaton $\mathcal{A}$ a regular expression α such that $\mathcal{L}(\alpha) = \mathcal{L}(\mathcal{A})$.

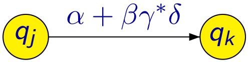
- We modify $\mathcal{A}$ in such a way that ensures it has exactly one initial and exactly one accepting state.
- Its states will be removed one by one.
- Its transitions will be labelled with regular expressions.
- The resulting automaton will have only two states – the initial and the accepting, and only one transition labelled with the resulting regular expression.

Transformation of an Automaton to a Regular Expression

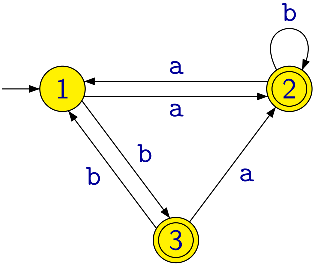
The main idea: If a state q is removed, for every pair of remaining states q_j, q_k we extend the label on a transition from q_j to q_k by a regular expression representing paths from q_j to q_k going through q.



After removing of the state q:



Example:



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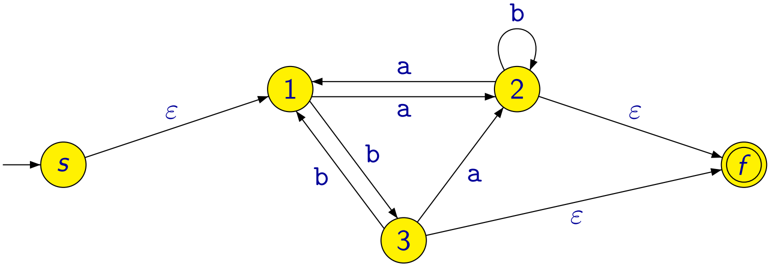

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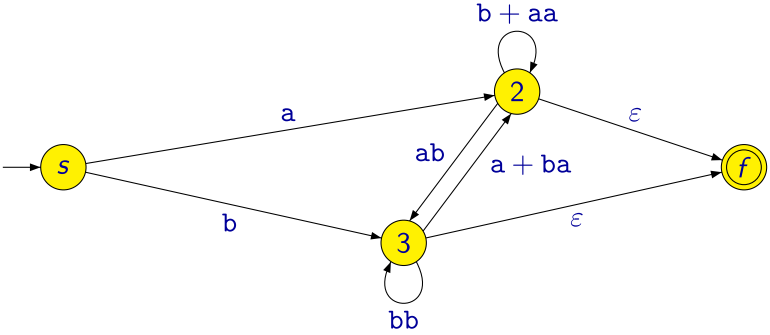

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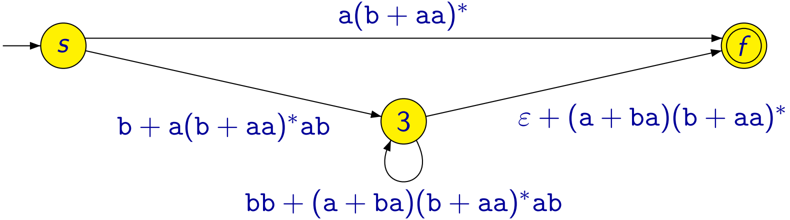

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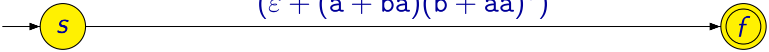

$$\mathtt{a}(\mathtt{b}+\mathtt{aa})^*+$$
$$(\mathtt{b}+\mathtt{a}(\mathtt{b}+\mathtt{aa})^*\mathtt{ab})$$
$$(\mathtt{bb}+(\mathtt{a}+\mathtt{ba})(\mathtt{b}+\mathtt{aa})^*\mathtt{ab})^*$$
$$(\varepsilon+(\mathtt{a}+\mathtt{ba})(\mathtt{b}+\mathtt{aa})^*)$$

Equivalence of Finite Automata and Regular Expressions

Theorem

A language is regular iff it can be represented by a regular expression.

Nonregular Languages

Nonregular Languages

Not all languages are regular.

There are languages for which there exist no finite automata accepting them.

Examples of nonregular languages:

- $L_1 = \{a^n b^n \mid n \geq 0\}$
- $L_2 = \{ww \mid w \in \{a, b\}^*\}$
- $L_3 = \{ww^R \mid w \in \{a, b\}^*\}$

Remark: The existence of nonregular languages is already apparent from the fact that there are only countably many (nonisomorphic) automata working over some alphabet Σ but there are uncountably many languages over the alphabet Σ.

Nonregular Languages

How to prove that some language L is not regular?

A language is not regular if there is no automaton (i.e., it is not possible to construct an automaton) accepting the language.

But how to prove that something does not exist?

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But how to prove that something does not exist?

The answer: By contradiction.

E.g., we can assume there is some automaton $\mathcal{A}$ accepting the language L, and show that this assumption leads to a contradiction.

Nonregular Languages

We show that language $L = \{a^n b^n \mid n \geq 0\}$ is not regular.

The proof by contradiction.

Let us assume there exists a DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ such that $\mathcal{L}(\mathcal{A}) = L$.

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Let $|Q| = n$.

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Consider word $z = a^n b^n$.

Since $z \in L$, there must be an accepting computation of the automaton $\mathcal{A}$

$$q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{a} \cdots \xrightarrow{a} q_{n-1} \xrightarrow{a} q_n \xrightarrow{b} q_{n+1} \xrightarrow{b} \cdots \xrightarrow{b} q_{2n-1} \xrightarrow{b} q_{2n}$$

where q_0 is an initial state, and $q_{2n} \in F$.

Nonregular Languages

Consider now the first $n+1$ states of the computation

$$q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{a} \cdots \xrightarrow{a} q_{n-1} \xrightarrow{a} q_n \xrightarrow{b} q_{n+1} \xrightarrow{b} \cdots \xrightarrow{b} q_{2n-1} \xrightarrow{b} q_{2n}$$

i.e., the sequence of states $q_0, q_1, \ldots, q_n$.

It is obvious that all states in this sequence can not be pairwise different, since $|Q| = n$ and the sequence has $n+1$ elements.

This means that there exists a state $q \in Q$ which occurs (at least) twice in the sequence.

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It is an application of so called **pigeonhole principle**.

Pigeonhole principle

If we have $n+1$ pigeons in n holes then there is at least one hole containing at least two pigeons.

Nonregular Languages

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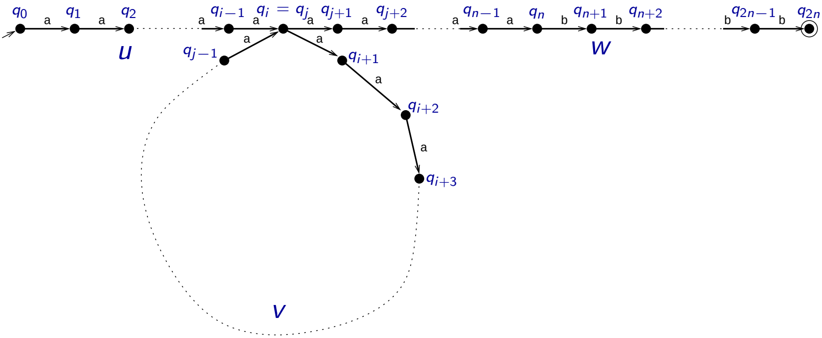
I.e., there are indexes i, j such that $0 \leq i < j \leq n$ and

$$q_i = q_j$$

which means that the automaton $\mathcal{A}$ must go through a cycle when reading the symbols a in the word $z = a^n b^n$.



The word $z = a^n b^n$ can be divided into three parts u, v, w such that $z = uvw$:

$$u = a^i \qquad v = a^{j-i} \qquad w = a^{n-j} b^n$$

Nonregular Languages

For the words $u = a^i$, $v = a^{j-i}$, and $w = a^{n-j}b^n$ we have

$$q_0 \xrightarrow{u} q_i \qquad q_i \xrightarrow{v} q_j \qquad q_j \xrightarrow{w} q_{2n}$$

Let r be the length of the word v, i.e., $r = j - i$ (obviously $r > 0$, due to $i < j$).

Since $q_i = q_j$, the automaton accepts word $uw = a^{n-r}b^n$ that does not belong to L:

$$q_0 \xrightarrow{u} q_i \xrightarrow{w} q_{2n}$$

The word $uvvw = a^{n+r}b^n$, that also does not belong to L, is accepted too:

$$q_0 \xrightarrow{u} q_i \xrightarrow{v} q_i \xrightarrow{v} q_i \xrightarrow{w} q_{2n}$$

Nonregular Languages

Similarly we can show that every word of the form $uvvvv \cdots vvw$, i.e., of the form $uv^k w$ for some $k \geq 0$, is accepted by the automaton $\mathcal{A}$:

$$q_0 \xrightarrow{u} q_i \xrightarrow{v} q_i \xrightarrow{v} q_i \xrightarrow{v} \cdots \xrightarrow{v} q_i \xrightarrow{v} q_i \xrightarrow{w} q_{2n}$$

A word of the form $uv^k w$ looks as follows: $a^{n-r+rk} b^n$.

Since $r > 0$, the following equivalence holds only for $k = 1$:

$$n - r + rk = n$$

This means that if $k \neq 1$ then $uv^k w$ does not belong to the language L. However, the automaton $\mathcal{A}$ accepts each such word, which is a contradiction with the assumption that $\mathcal{L}(\mathcal{A}) = \{a^n b^n \mid n \geq 0\}$.

Pumping Lemma

Let us assume that language L is accepted by some particular automaton $\mathcal{A}$, i.e., $L = \mathcal{L}(\mathcal{A})$.

Let us consider some arbitrary word $z \in L$ where $z = a_1 a_2 \cdots a_k$.

Since automaton $\mathcal{A}$ accepts word z, there must be some accepting computation of the automaton, i.e., a sequence of states:

$$q_0,\ q_1,\ q_2,\ \ldots,\ q_{k-1},\ q_k$$

of length $k + 1$ where

- q_0 is an initial state
- $q_{i-1} \xrightarrow{a_i} q_i$ for each $i \in \{1, 2, \ldots, k\}$
- q_k is an accepting state

Pumping Lemma

Let us assume that $\mathcal{A}$ has n states (i.e., $|Q| = n$), and that $|z| \geq n$.

Since $|z| = k$, the computation of automaton $\mathcal{A}$ over word z forms a sequence, whose length is at least $n+1$, that contains at most n different states:

$$q_0,\ q_1,\ q_2,\ \ldots,\ q_{k-1},\ q_k$$

It follows that there must be at least one state q that occurs at least twice in this sequence (recall the *pigeonhole principle*).

Pumping Lemma

Let us say that the repeated state occurs on positions i and j, i.e., $q_i = q_j$ where $i < j$.

$$q_0, \cdots, q_i, \cdots, q_j, \cdots, q_k$$

Remark: It is obvious that in fact we can find i and j such that $i < j \leq n$.

The word z can be divided into three parts:

$$\underbrace{a_1 \cdots a_i}_{u} \quad \underbrace{a_{i+1} \cdots a_j}_{v} \quad \underbrace{a_{j+1} \cdots a_k}_{w}$$

- $q_0 \xrightarrow{u} q_i$
- $q_i \xrightarrow{v} q_j$ (and so also $q_i \xrightarrow{v} q_i$ since $q_j = q_i$)
- $q_j \xrightarrow{w} q_k$ (and so also $q_i \xrightarrow{w} q_k$ since $q_j = q_i$)

Pumping Lemma

Consider now words:

$$\underbrace{a_1 \cdots a_i}_{u} \quad \underbrace{a_{j+1} \cdots a_k}_{w}$$

$$\underbrace{a_1 \cdots a_i}_{u} \quad \underbrace{a_{i+1} \cdots a_j}_{v} \quad \underbrace{a_{i+1} \cdots a_j}_{v} \quad \underbrace{a_{j+1} \cdots a_k}_{w}$$

$$\underbrace{a_1 \cdots a_i}_{u} \quad \underbrace{a_{i+1} \cdots a_j}_{v} \quad \underbrace{a_{i+1} \cdots a_j}_{v} \quad \underbrace{a_{i+1} \cdots a_j}_{v} \quad \underbrace{a_{j+1} \cdots a_k}_{w}$$

$$\cdots$$

It is obvious that A accepts all of them because

- $q_0 \xrightarrow{u} q_i$
- $q_i \xrightarrow{v} q_i$
- $q_i \xrightarrow{w} q_k$ where $q_k \in F$

Pumping Lemma

Pumping Lemma

If language L is regular then there exists $n \in \mathbb{N}$ such that every word $z \in L$ such that $|z| \geq n$ can be divided into subwords u, v, w such that $z = uvw$, $|uv| \leq n$, $|v| \geq 1$, and for every $i \geq 0$ it holds that $uv^i w \in L$.

Formally:

If L is regular then

$$(\exists n \in \mathbb{N})(\forall z \in L \text{ s.t. } |z| \geq n)(\exists u, v, w \text{ s.t. } z = uvw, |uv| \leq n, |v| \geq 1)$$
$$(\forall i \geq 0) : uv^i w \in L$$

Pumping Lemma

We can take the contrapositive of the pumping lemma. ($A \Rightarrow B$ is equivalent to $\neg B \Rightarrow \neg A$.)

If

$$(\forall n \in \mathbb{N})(\exists z \in L \text{ s.t. } |z| \geq n)(\forall u, v, w \text{ s.t. } z = uvw,\ |uv| \leq n,\ |v| \geq 1)$$
$$(\exists i \geq 0) : uv^i w \notin L,$$

then L is not regular.

So if we want to show that a language L is not regular, it is sufficient to show that L satisfies this condition.

Example: Let us consider laguage $L = \{a^i b^i \mid i \geq 0\}$.

- Let us assume that L is accepted by some automaton with n states.
- Let us consider word $z = a^n b^n$.
- Let us consider all possibilities how z can be divided into three subwords u, v, w satisfying conditions $|uv| \leq n$ and $|v| \geq 1$.

  It is obvious that words u and v contain only symbols a. For every particular division there are some j and k such that $j + k \leq n$, $k \geq 1$, and
  - $u = a^j$
  - $v = a^k$
  - $w = a^{n-(j+k)} b^n$
- If we choose $i = 0$, we obtain $uv^i w = uw = a^{n-k} b^n$. Since $n - k < n$, we have $uv^i w \notin L$.

Remark: Proving that some first order logic formula with alternating universal and existential quantifiers can be viewed as game played by two players, Player A and Player B.

Player A chooses values of variables bound by existential quantifiers and Player B values of variables bound by universal quantifiers.

If we want to refute the given claim, it is sufficient to find a winning strategy for Player B.

If L is regular then
$(\exists n \in \mathbb{N})(\forall z \in L \text{ s.t. } |z| \geq n)(\exists u, v, w \text{ s.t. } z = uvw,\ |uv| \leq n,\ |v| \geq 1)$
$(\forall i \geq 0) : uv^i w \in L$.

The game for Pumping Lemma looks as follows:

1. Player A chooses some $n \in \mathbb{N}$.
2. Player B chooses a word z such that $z \in L$ and $|z| \geq n$.
3. Player A chooses words u, v, w such that $z = uvw$, $|uv| \leq n$, $|v| \geq 1$.
4. Player B chooses $i \geq 0$.
5. If $uv^i w \in L$ then Player A wins. If $uv^i w \notin L$ then Player B wins.

If Player B has a winning strategy in this game then L is not regular.

Pumping Lemma

Example: $L = \{a^i b^i \mid i \geq 0\}$

1. Player A chooses $n > 0$.
2. Player B chooses $z = a^n b^n$.
3. Player A chooses words u, v, w such that $z = uvw$, $|uv| \leq n$, $|v| \geq 1$.
4. Player B chooses $i = 0$.
5. Player B wins, since no matter what Player A does, we always have $uv^i w \notin L$ because a non-empty word z occurs in the part of word z consisting only of symbols a, and when we omit it, we obtain a word of the form $a^k b^n$ where $k < n$, which does not belong to L.