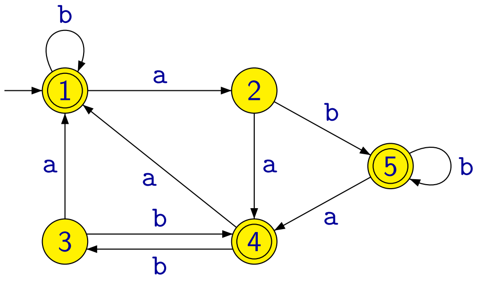
Finite Automata

Deterministic Finite Automaton



A **deterministic finite automaton** consists of **states** and **transitions**. One of the states is denoted as an **initial state** and some of states are denoted as **accepting**.

Deterministic Finite Automaton

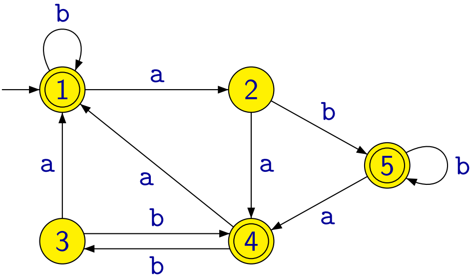
Formally, a **deterministic finite automaton** (**DFA**) is defined as a tuple

$$(Q, \Sigma, \delta, q_0, F)$$

where:

- Q is a nonempty finite set of **states**
- Σ is an **alphabet** (a nonempty finite set of symbols)
- $\delta : Q \times \Sigma \to Q$ is a **transition function**
- $q_0 \in Q$ is an **initial state**
- $F \subseteq Q$ is a set of **accepting states**

Deterministic Finite Automaton



- $Q = \{1, 2, 3, 4, 5\}$
- $\Sigma = \{\mathtt{a}, \mathtt{b}\}$
- $q_0 = 1$
- $F = \{1, 4, 5\}$

$$
\begin{array}{ll}
\delta(1, \mathtt{a}) = 2 & \delta(1, \mathtt{b}) = 1 \\
\delta(2, \mathtt{a}) = 4 & \delta(2, \mathtt{b}) = 5 \\
\delta(3, \mathtt{a}) = 1 & \delta(3, \mathtt{b}) = 4 \\
\delta(4, \mathtt{a}) = 1 & \delta(4, \mathtt{b}) = 3 \\
\delta(5, \mathtt{a}) = 4 & \delta(5, \mathtt{b}) = 5
\end{array}
$$

Deterministic Finite Automaton

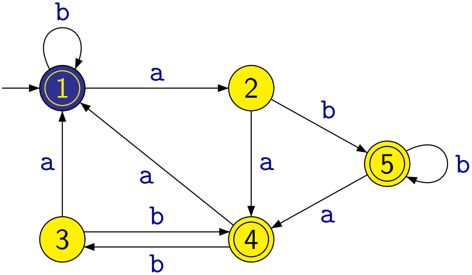
Instead of

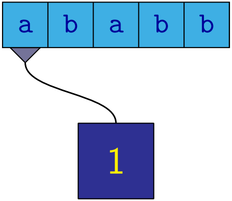
$$\begin{array}{ll} \delta(1,\mathtt{a}) = 2 & \delta(1,\mathtt{b}) = 1 \\ \delta(2,\mathtt{a}) = 4 & \delta(2,\mathtt{b}) = 5 \\ \delta(3,\mathtt{a}) = 1 & \delta(3,\mathtt{b}) = 4 \\ \delta(4,\mathtt{a}) = 1 & \delta(4,\mathtt{b}) = 3 \\ \delta(5,\mathtt{a}) = 4 & \delta(5,\mathtt{b}) = 5 \end{array}$$

we rather use a more succinct representation as a table or a depicted graph:

δ	a	b
$\leftrightarrow 1$	2	1
2	4	5
3	1	4
$\leftarrow 4$	1	3
$\leftarrow 5$	4	5

Deterministic Finite Automaton





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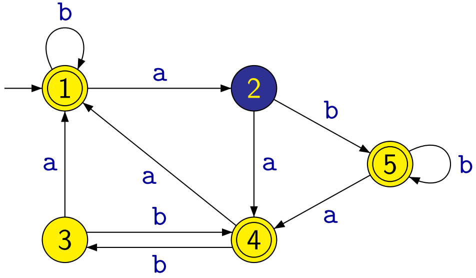
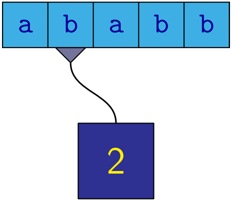
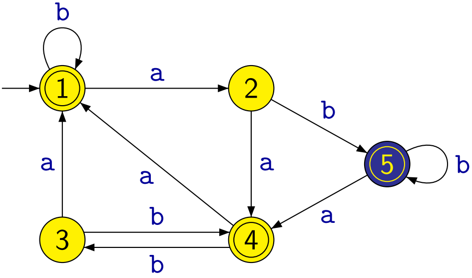
Deterministic Finite Automaton

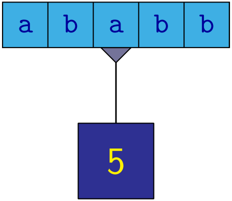




$$1 \xrightarrow{a} 2$$

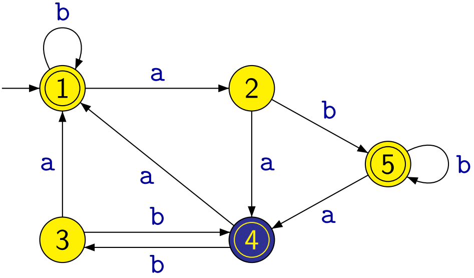
Deterministic Finite Automaton

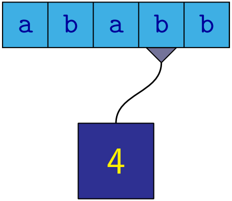




$$1 \xrightarrow{a} 2 \xrightarrow{b} 5$$

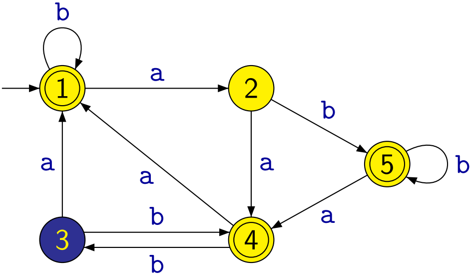
Deterministic Finite Automaton

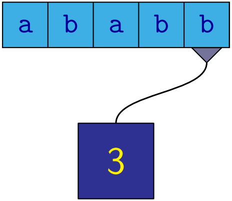




$$1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{b}} 5 \xrightarrow{\texttt{a}} 4$$

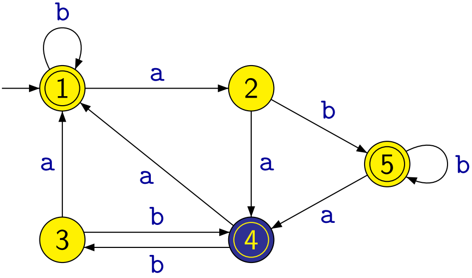
Deterministic Finite Automaton

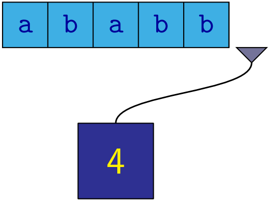




$$1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{b}} 5 \xrightarrow{\texttt{a}} 4 \xrightarrow{\texttt{b}} 3$$

Deterministic Finite Automaton





$1 \xrightarrow{a} 2 \xrightarrow{b} 5 \xrightarrow{a} 4 \xrightarrow{b} 3 \xrightarrow{b} 4$

Deterministic Finite Automaton

Definition

Let us have a DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$.

By $q \xrightarrow{w} q'$, where $q, q' \in Q$ and $w \in \Sigma^*$, we denote the fact that the automaton, starting in state q goes to state q' by reading word w.

Remark: $\longrightarrow \subseteq Q \times \Sigma^* \times Q$ is a ternary relation.

Instead of $(q, w, q') \in \longrightarrow$ we write $q \xrightarrow{w} q'$.

It holds for a DFA that for each state q and each word w there is exactly one state q' such that $q \xrightarrow{w} q'$.

Deterministic Finite Automaton

Relation $\longrightarrow$ can be formally defined by the following inductive definition:

- $q \xrightarrow{\varepsilon} q$ for each $q \in Q$
- For $w \in \Sigma^*$ and $a \in \Sigma$:

  $q \xrightarrow{wa} q'$ iff there is $q'' \in Q$ such that

  $$q \xrightarrow{w} q'' \text{ and } \delta(q'', a) = q'$$

Deterministic Finite Automaton

	a	b
$\leftrightarrow$ 1	2	1
2	4	5
3	1	4
$\leftarrow$ 4	1	3
$\leftarrow$ 5	4	5

$1 \xrightarrow{\varepsilon} 1$ $\quad$ $\delta(1, \mathtt{a}) = 2$

$1 \xrightarrow{\mathtt{a}} 2$ $\quad$ $\delta(2, \mathtt{b}) = 5$

$1 \xrightarrow{\mathtt{ab}} 5$ $\quad$ $\delta(5, \mathtt{a}) = 4$

$1 \xrightarrow{\mathtt{aba}} 4$ $\quad$ $\delta(4, \mathtt{b}) = 3$

$1 \xrightarrow{\mathtt{abab}} 3$ $\quad$ $\delta(3, \mathtt{b}) = 4$

$1 \xrightarrow{\mathtt{ababb}} 4$

Deterministic Finite Automaton

A word $w \in \Sigma^*$ is **accepted** by a deterministic finite automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ iff there exists a state $q \in F$ such that $q_0 \xrightarrow{w} q$.

Definition

A **language** accepted by a given deterministic finite automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, denoted $\mathcal{L}(\mathcal{A})$, is the set of all words accepted by the automaton, i.e.,

$$\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^* \mid \exists q \in F : q_0 \xrightarrow{w} q\}$$

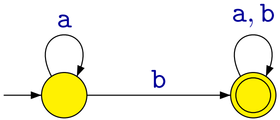
Regular languages

Definition

A language L is **regular** iff there exists some deterministic finite automaton accepting L, i.e., DFA $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = L$.

Example: An automaton recognizing the language L over alphabet $\{\mathtt{a}, \mathtt{b}\}$ consisting of those words that contain at least one occurrence of symbol $\mathtt{b}$, i.e.,

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid |w|_{\mathtt{b}} \geq 1\}$$



Examples of Deterministic Finite Automata

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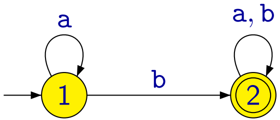
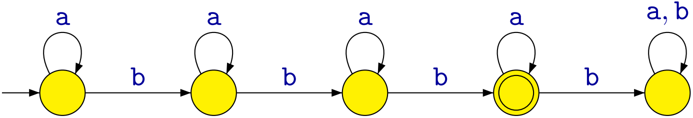


	a	b
$\rightarrow 1$	1	2
$\leftarrow 2$	2	2

Examples of Deterministic Finite Automata

Example: An automaton recognizing the language L over alphabet $\{\mathtt{a}, \mathtt{b}\}$ consisting of those words that contain exactly three occurrences of symbol b, i.e.,

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid |w|_{\mathtt{b}} = 3\}$$



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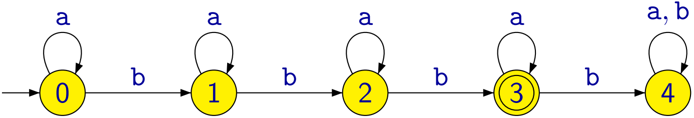
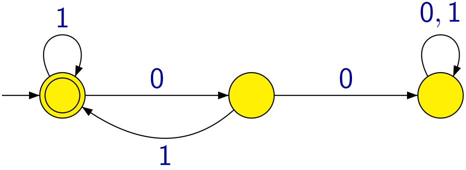


	a	b
$\rightarrow$ 0	0	1
1	1	2
2	2	3
$\leftarrow$ 3	3	4
4	4	4

Example: An automaton recognizing the language over alphabet $\{0, 1\}$ consisting of those words where every occurrence of symbol 0 is immediately followed with symbol 1.



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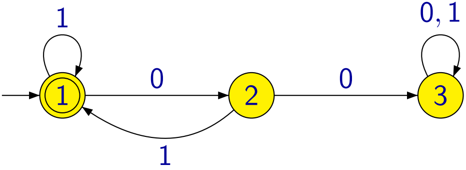
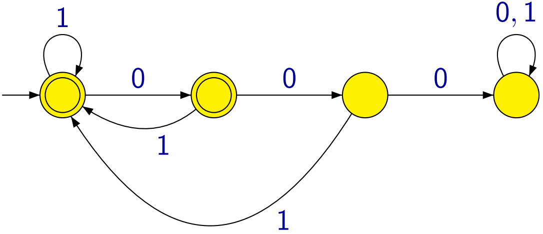


	0	1
$\leftrightarrow$ 1	2	1
2	3	1
3	3	3

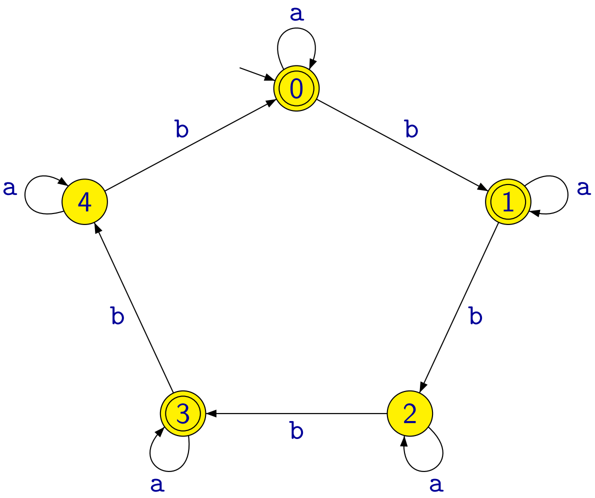
Example: An automaton recognizing the language over alphabet $\{0, 1\}$ consisting of those words where every pair of consecutive symbols 0 is immediately followed with symbol 1.



Examples of Deterministic Finite Automata

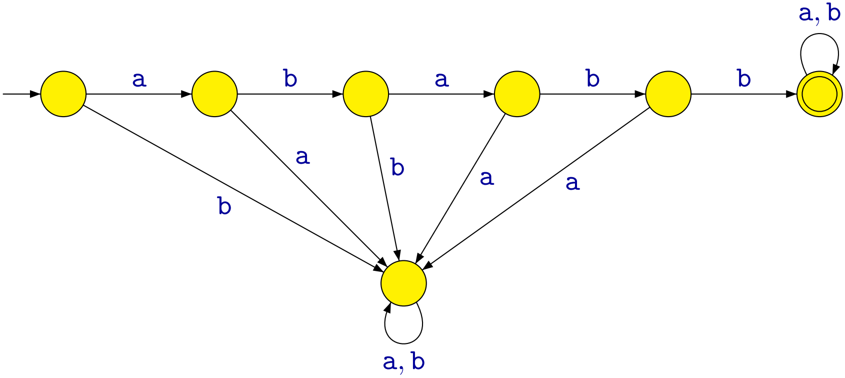
Example: An automaton recognizing the language

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid (|w|_{\mathtt{b}} \bmod 5) \in \{0, 1, 3\}\}$$

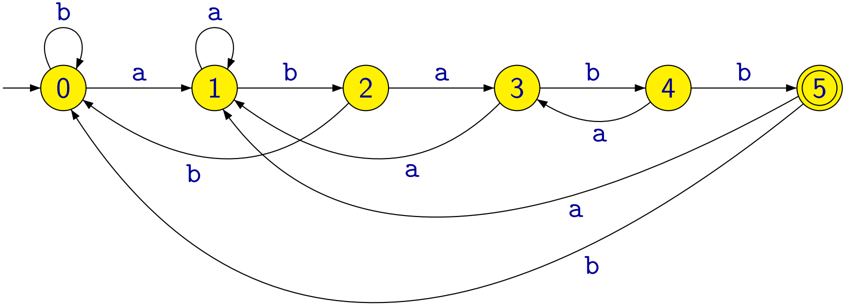


Examples of Deterministic Finite Automata

Example: An automaton recognizing the language over alphabet $\{a, b\}$ consisting of those words that start with the **prefix** ababb.



Example: An automaton recognizing the language over alphabet $\{a, b\}$ of those words that end with **suffix** ababb.



Examples of Deterministic Finite Automata

The construction of this automaton is based on the following idea:

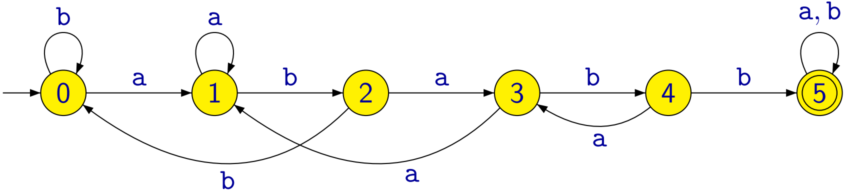
- Let us assume that we want to search for a word u of length n (i.e., $|u| = n$).
  The states of the automaton are denoted with numbers $0, 1, \ldots, n$.
- A state with number i corresponds to the situation when i is the length of the longest word that is at the same time:
  - a prefix of the pattern u we are searching for
  - a suffix of the part of the input word that the automaton has read so far

For example, for the searched pattern `ababb` the states of the automaton correspond to the following words:

- State 0 ... ε
- State 1 ... `a`
- State 2 ... `ab`
- State 3 ... `aba`
- State 4 ... `abab`
- State 5 ... `ababb`

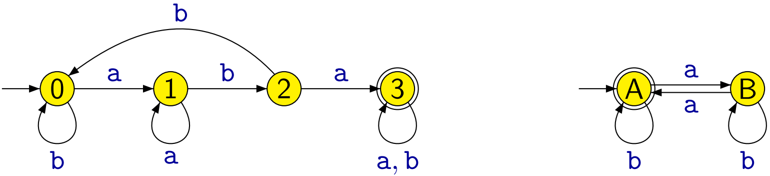
Examples of Deterministic Finite Automata

Example: An automaton recognizing the language over alphabet $\{\mathtt{a}, \mathtt{b}\}$ consisting of those words that contain **subword** `ababb`.



An Automaton for Intersection of Languages

Let us have the following two automata:



Do both of them accept the word ababb?

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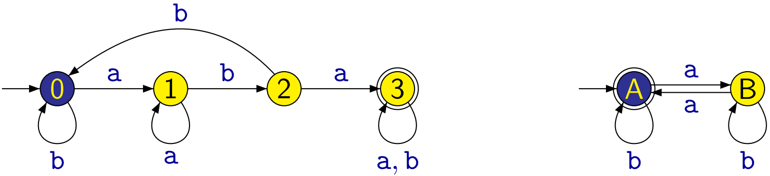

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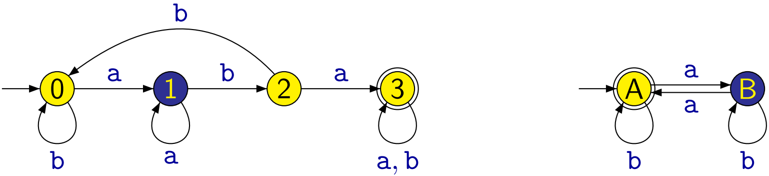

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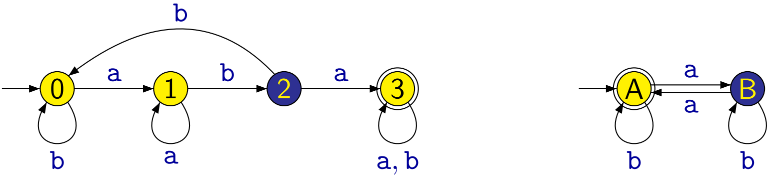

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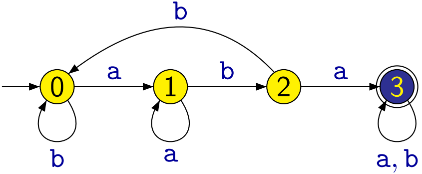

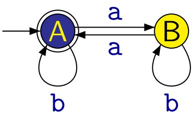

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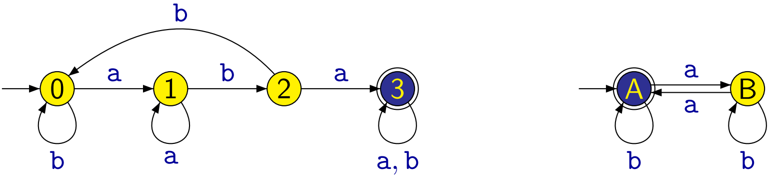

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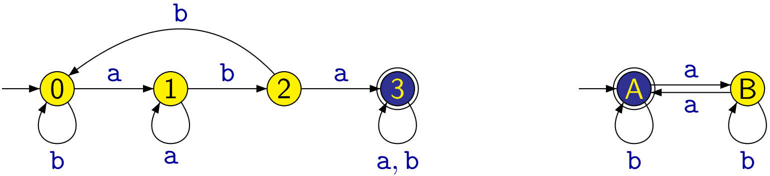

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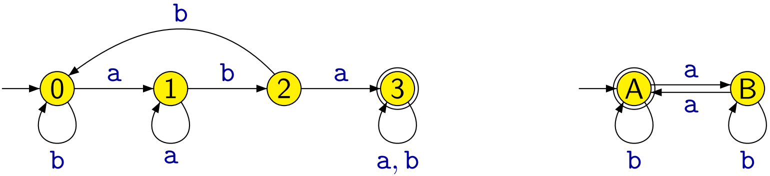

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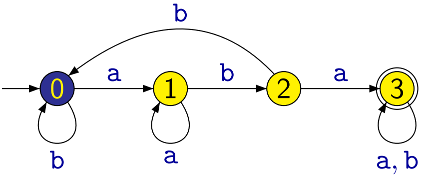

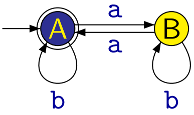

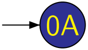

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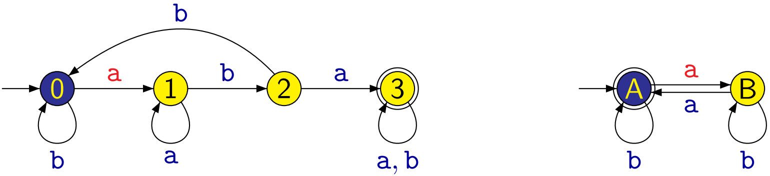

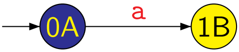

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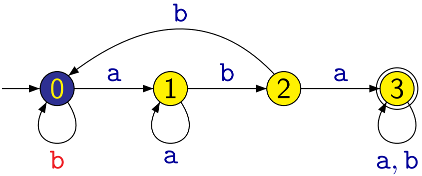

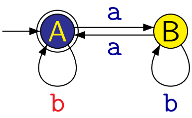

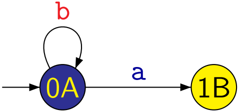

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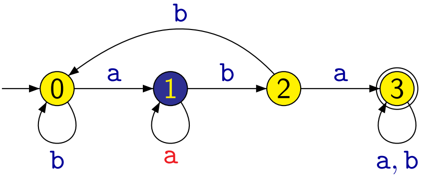

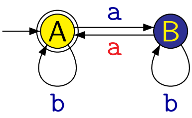

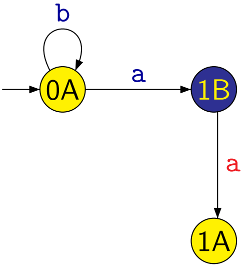

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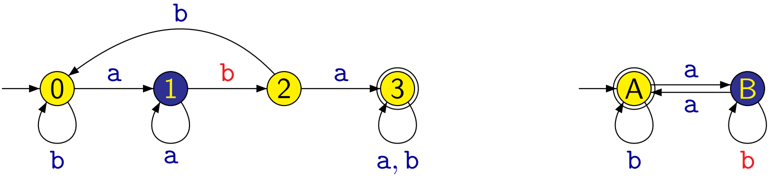

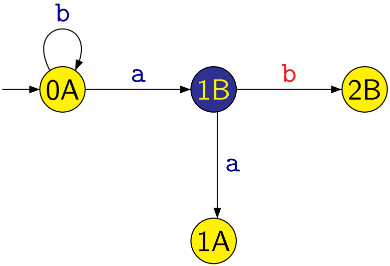

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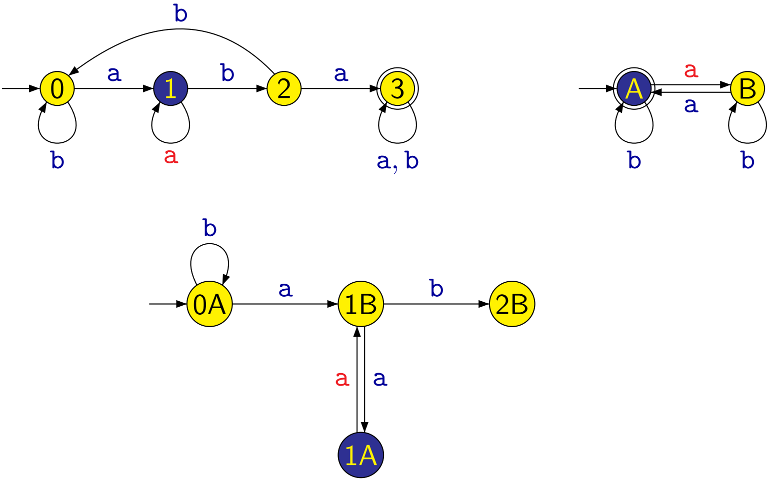

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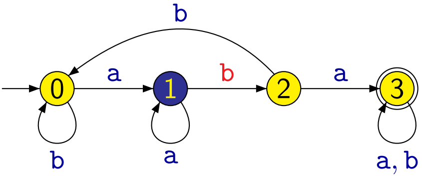
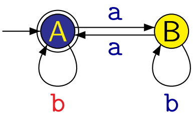
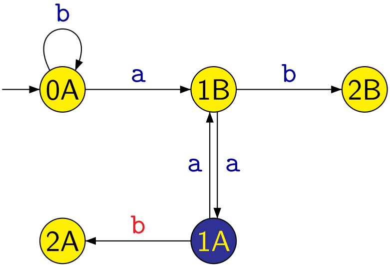
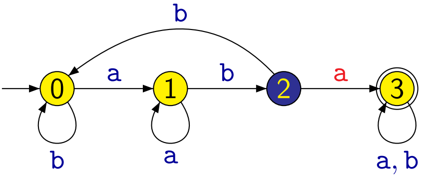
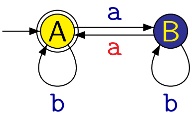


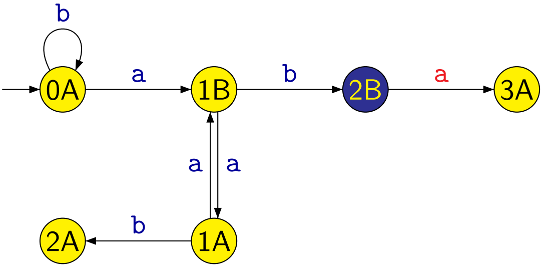




An Automaton for Intersection of Languages







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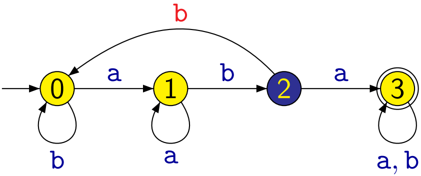

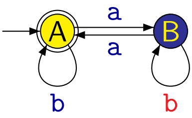

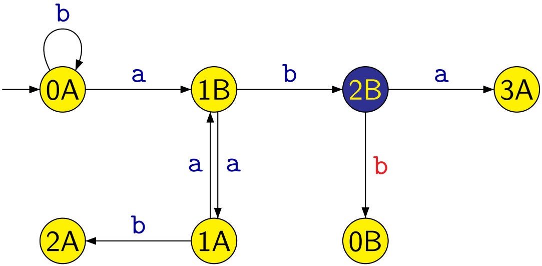

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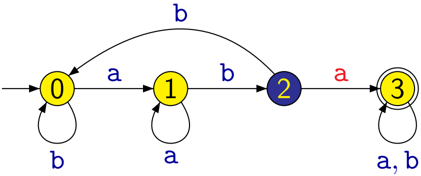

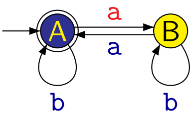

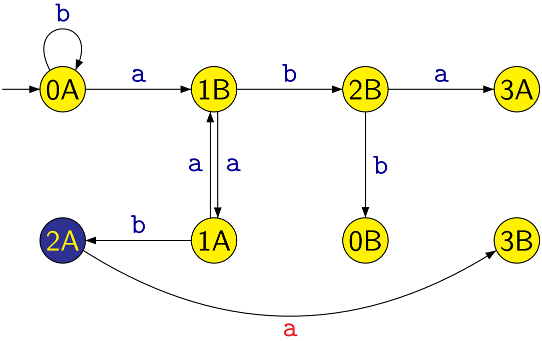

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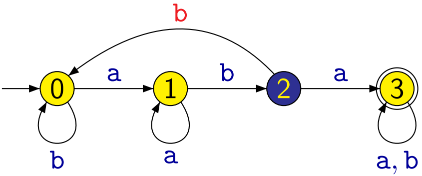

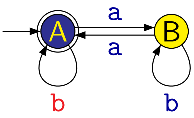

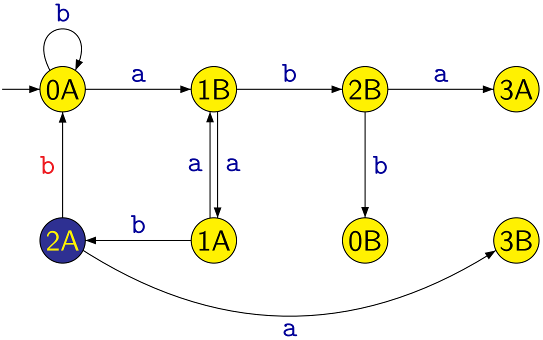

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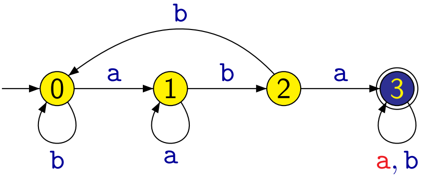
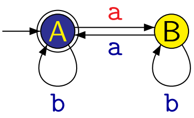
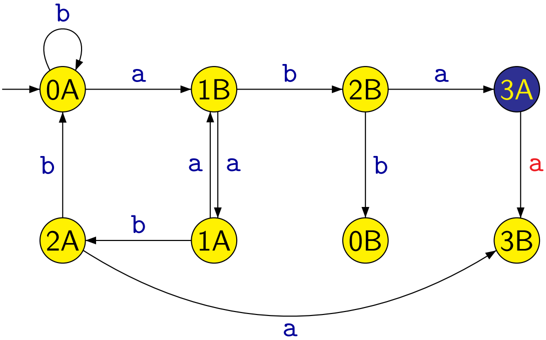
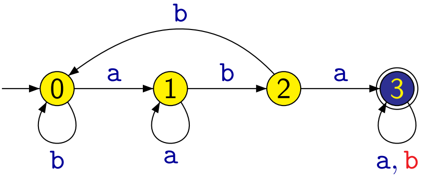
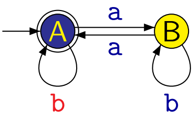


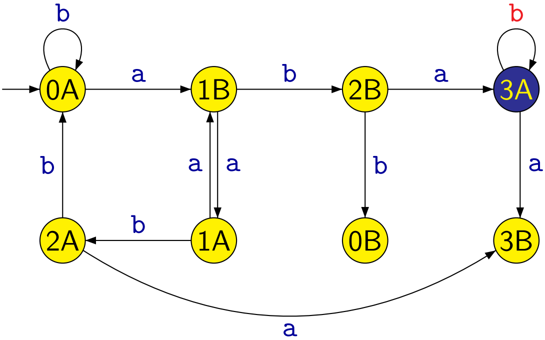




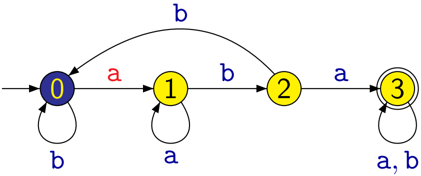
An Automaton for Intersection of Languages

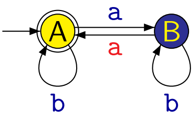


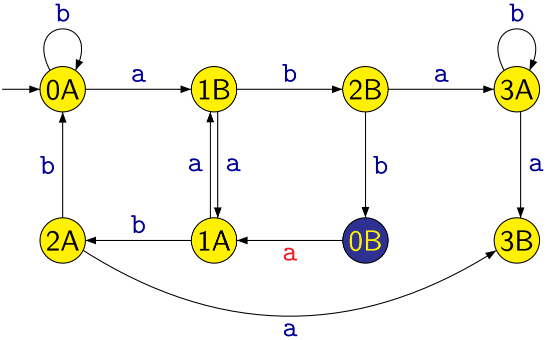




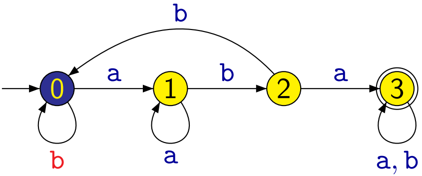
An Automaton for Intersection of Languages

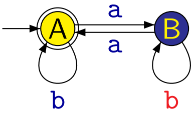


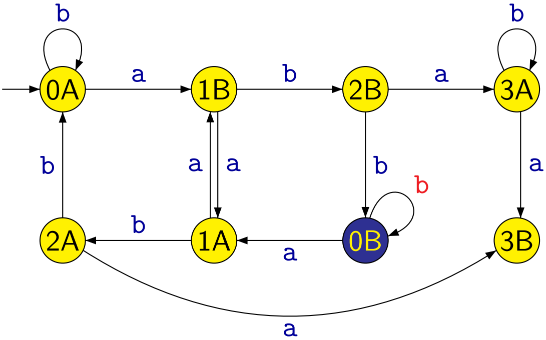




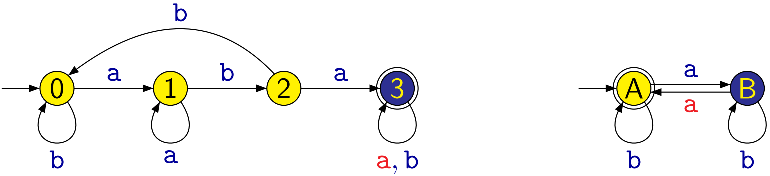
An Automaton for Intersection of Languages

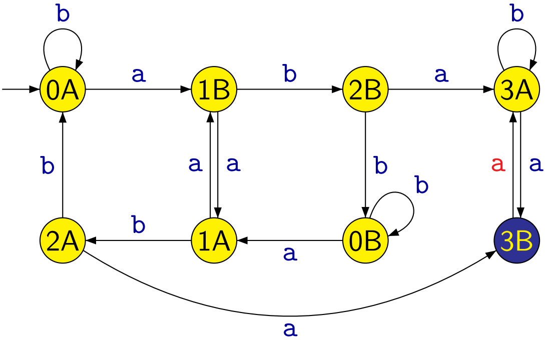






An Automaton for Intersection of Languages





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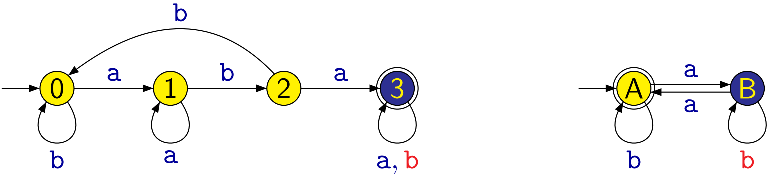
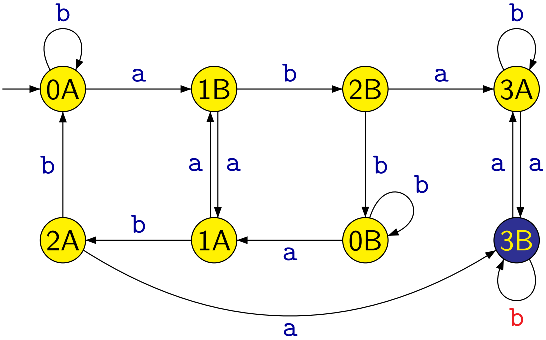
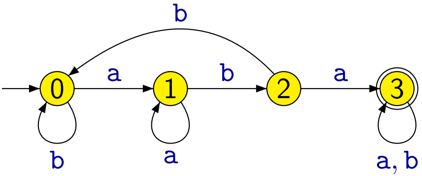
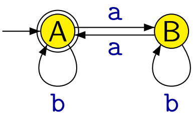
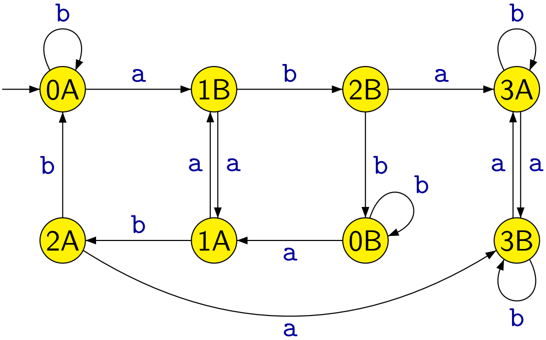




An Automaton for Intersection of Languages







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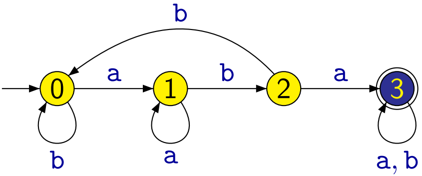
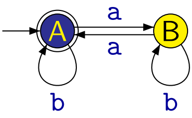
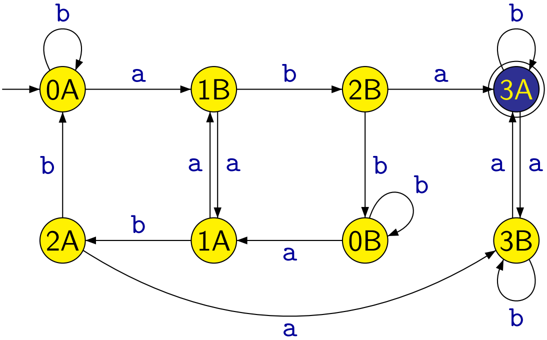
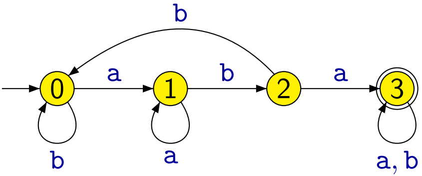
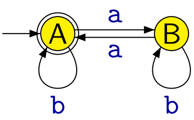


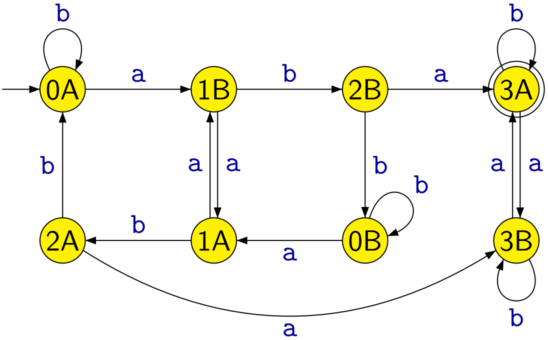




An Automaton for Intersection of Languages







An Automaton for Intersection of Languages

Formally, the construction can be described as follows:

We assume we have two deterministic finite automata $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$.

We construct DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ where:

- $Q = Q_1 \times Q_2$
- $\delta(\,(q_1, q_2), a\,) = (\,\delta_1(q_1, a), \delta_2(q_2, a)\,)$ for each $q_1 \in Q_1$, $q_2 \in Q_2$, $a \in \Sigma$
- $q_0 = (q_{01}, q_{02})$
- $F = F_1 \times F_2$

It is not difficult to check that for each word $w \in \Sigma^*$ we have $w \in \mathcal{L}(\mathcal{A})$ iff $w \in \mathcal{L}(\mathcal{A}_1)$ and $w \in \mathcal{L}(\mathcal{A}_2)$, i.e.,

$$\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$$

Intersection of Regular Languages

Theorem

If languages $L_1, L_2 \subseteq \Sigma^*$ are regular then also the language $L_1 \cap L_2$ is regular.

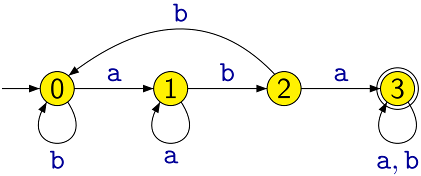
Proof: Let us assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are deterministic finite automata such that

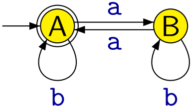
$$L_1 = \mathcal{L}(\mathcal{A}_1) \qquad L_2 = \mathcal{L}(\mathcal{A}_2)$$

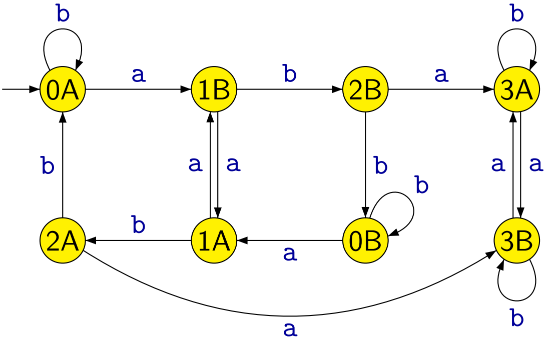
Using the described construction, we can construct a deterministic finite automaton $\mathcal{A}$ such that

$$\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2) = L_1 \cap L_2$$

An Automaton for the Union of Languages







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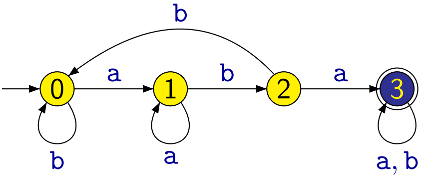

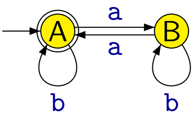

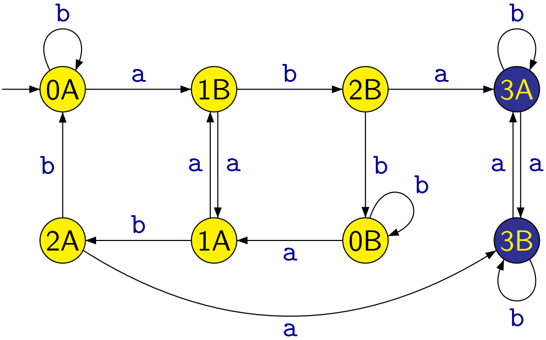

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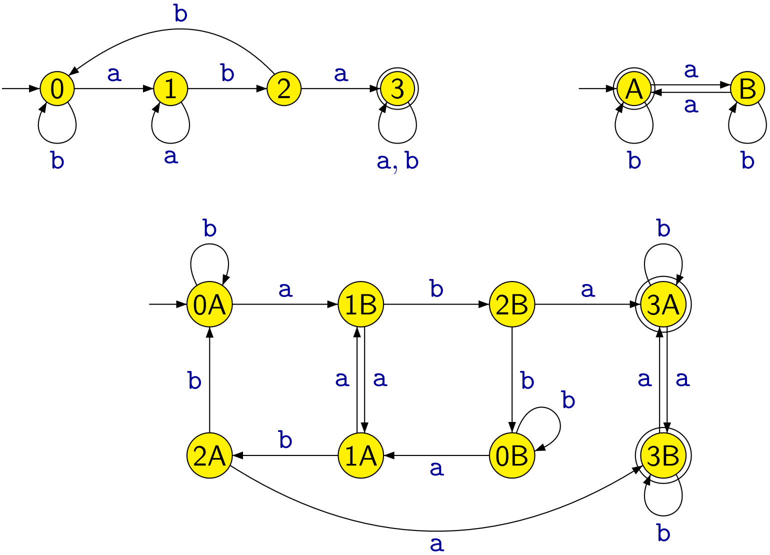

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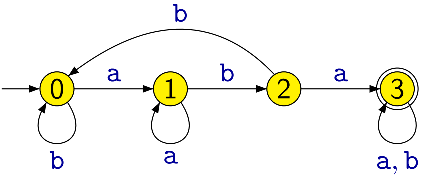

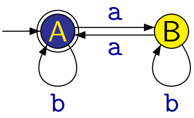

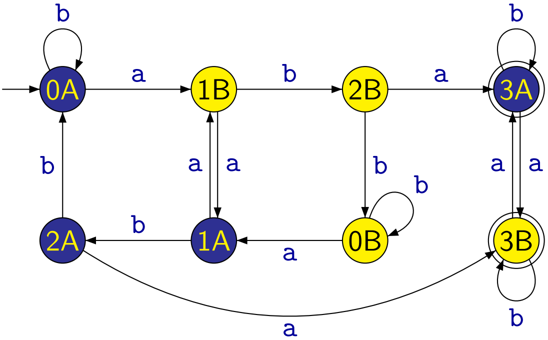

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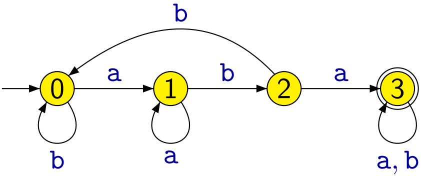

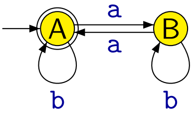

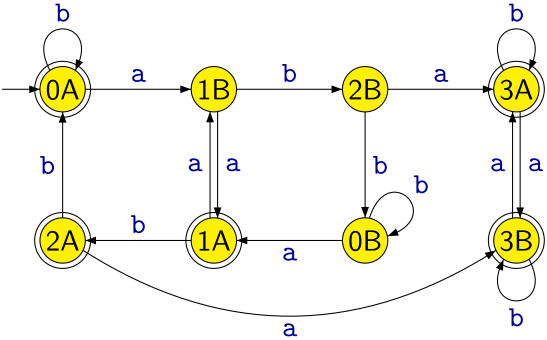

Union of Regular Languages

The construction of an automaton $\mathcal{A}$ that accepts the **union** of languages accepted by automata $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e., the language

$$\mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_1)$$

is almost identical as in the case of the automaton accepting $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$.

The only difference is the set of accepting states:

- $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

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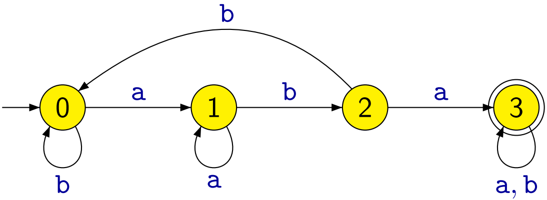
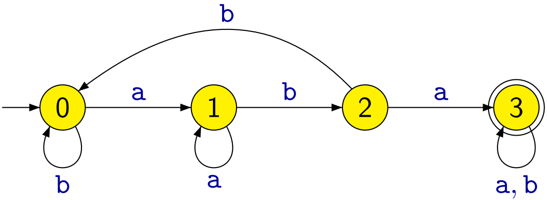
Theorem

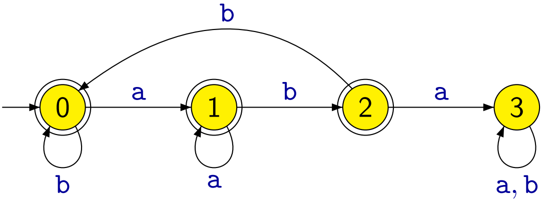
If languages $L_1, L_2 \subseteq \Sigma^*$ are regular then also the language $L_1 \cup L_2$ is regular.

An Automaton for the Complement of a Language



An Automaton for the Complement of a Language





Complement of a Regular Language

Given a DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ we construct DFA $\mathcal{A}' = (Q, \Sigma, \delta, q_0, Q - F)$.

It is obvious that for each word $w \in \Sigma^*$ we have $w \in \mathcal{L}(\mathcal{A}')$ iff $w \notin \mathcal{L}(\mathcal{A})$, i.e.,

$$\mathcal{L}(\mathcal{A}') = \overline{\mathcal{L}(\mathcal{A})}$$

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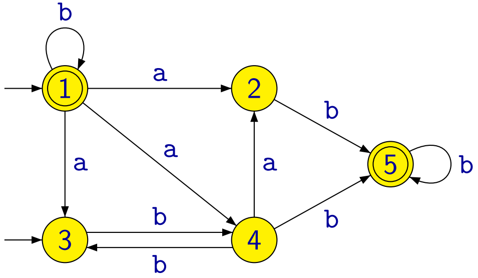
Theorem

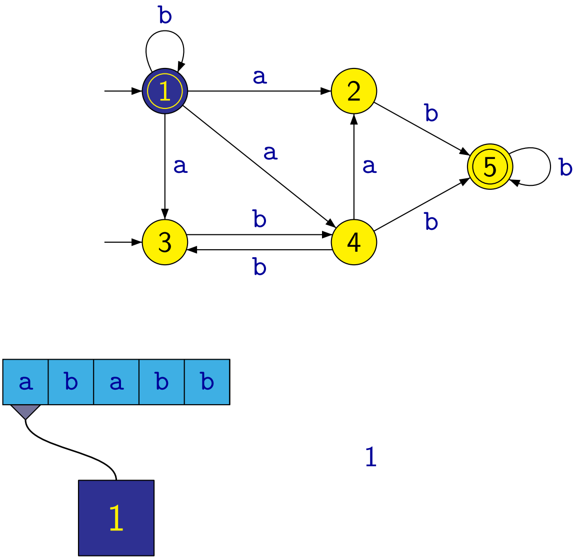
If a language L is regular then also its complement $\overline{L}$ is regular.

Nondeterministic Finite Automaton



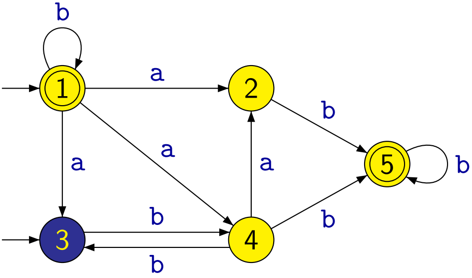
- The number of transitions going from one state and labelled with the same symbol can be arbitrary (including zero).
- There can be more than one initial state in the automaton.

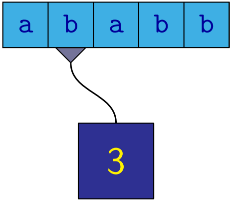
Nondeterministic Finite Automaton



1

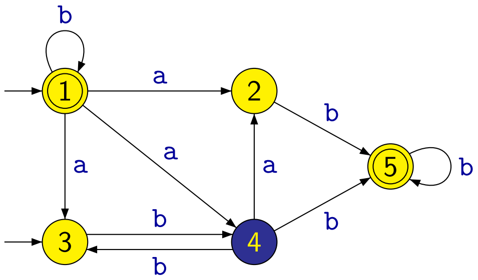
Nondeterministic Finite Automaton

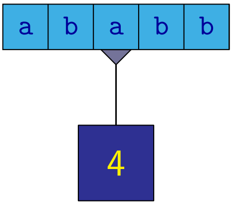




$1 \xrightarrow{a} 3$

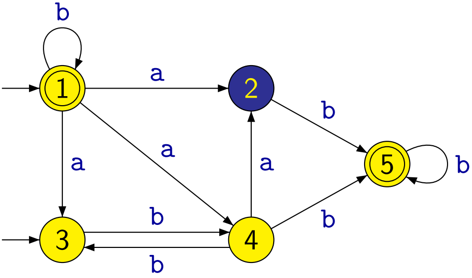
Nondeterministic Finite Automaton

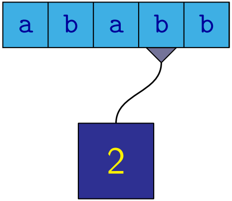




$1 \xrightarrow{a} 3 \xrightarrow{b} 4$

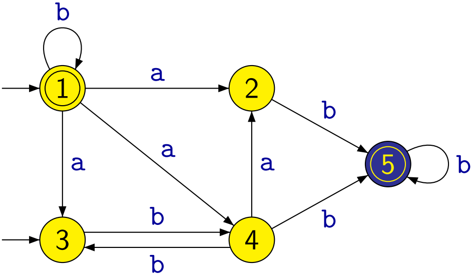
Nondeterministic Finite Automaton

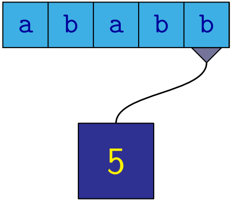




$$1 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 4 \xrightarrow{\texttt{a}} 2$$

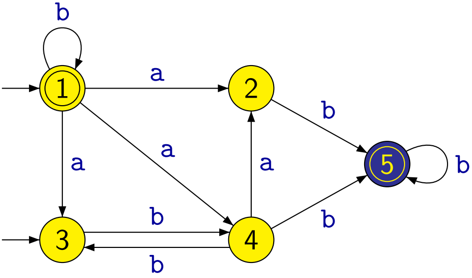
Nondeterministic Finite Automaton

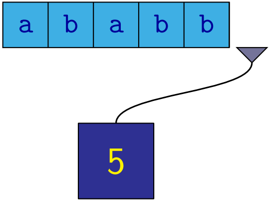




$$1 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{a} 2 \xrightarrow{b} 5$$

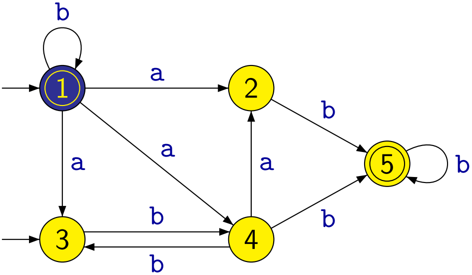
Nondeterministic Finite Automaton

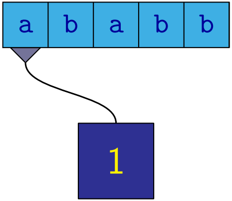




$1 \xrightarrow{a} 3 \xrightarrow{b} 4 \xrightarrow{a} 2 \xrightarrow{b} 5 \xrightarrow{b} 5$

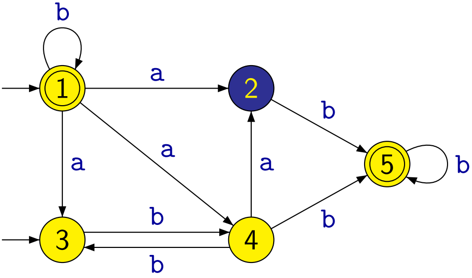
Nondeterministic Finite Automaton

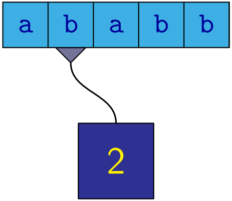




1

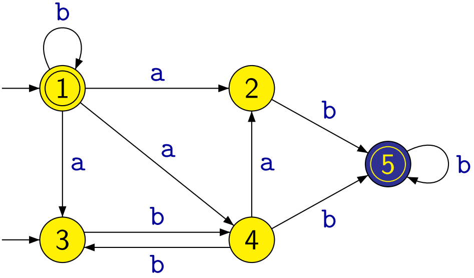
Nondeterministic Finite Automaton

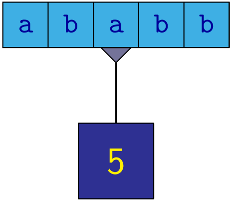




$1 \xrightarrow{a} 2$

Nondeterministic Finite Automaton

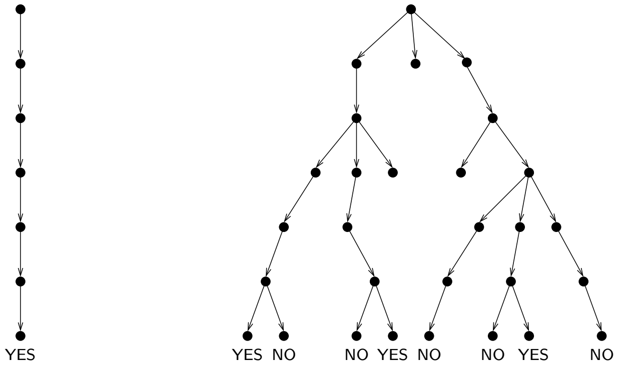




$$1 \xrightarrow{\mathtt{a}} 2 \xrightarrow{\mathtt{b}} 5$$

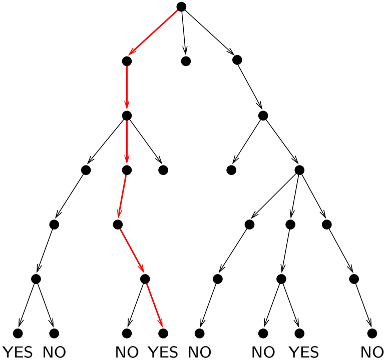
Nondeterministic Finite Automaton

A nondeterministic finite automaton accepts a given word if there **exists** at least one computation of the automaton that accepts the word.



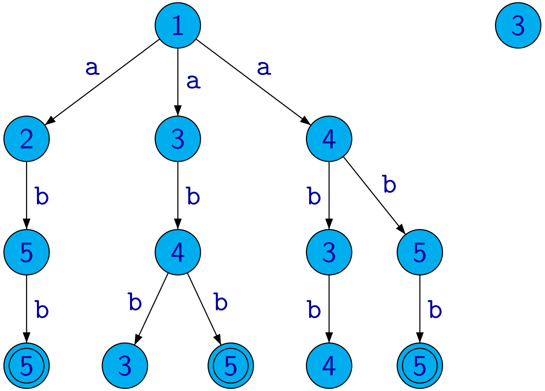
Nondeterministic Finite Automaton

A nondeterministic finite automaton accepts a given word if there **exists** at least one computation of the automaton that accepts the word.



Nondeterministic Finite Automaton

	a	b
$\leftrightarrow 1$	$2, 3, 4$	1
2	$-$	5
$\rightarrow 3$	$-$	4
4	2	$3, 5$
$\leftarrow 5$	$-$	5



Example: A forest representing all possible computations over the word abb.

Nondeterministic Finite Automaton

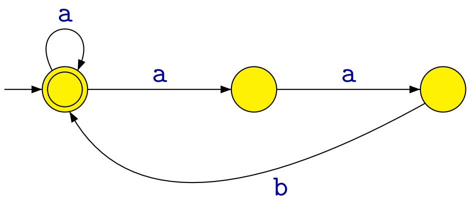
Formally, a **nondeterministic finite automaton** (**NFA**) is defined as a tuple

$$(Q, \Sigma, \delta, I, F)$$

where:

- Q is a finite set of **states**
- Σ is a finite **alphabet**
- $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$ is a **transition fuction**
- $I \subseteq Q$ is a set of **initial states**
- $F \subseteq Q$ is a set of **accepting states**

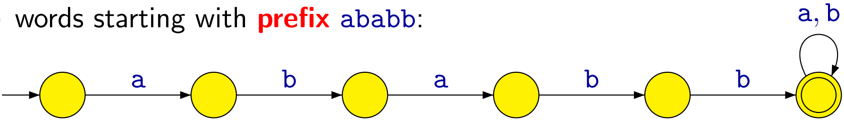
Example: An automaton recognizing the language over alphabet $\{a, b\}$ consisting of those words where every occurrence of symbol b is immediately preceded with two symbols a.



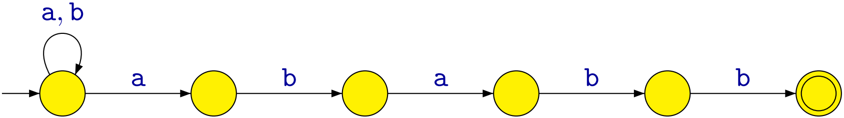
Examples of Nondeterministic Finite Automata

Example: An automaton recognizing the language over alphabet $\{a, b\}$:

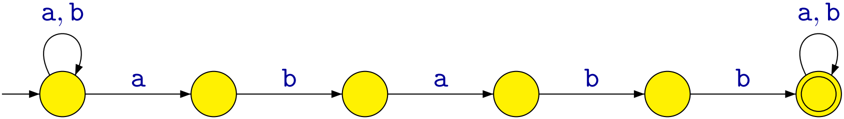
- words starting with **prefix** ababb:



- words ending with **suffix** ababb:

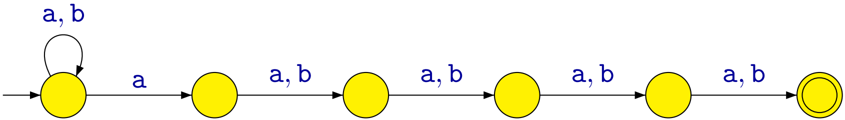


- words containing **subword** ababb:

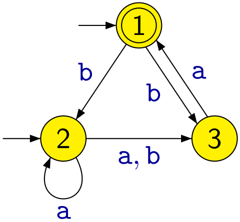


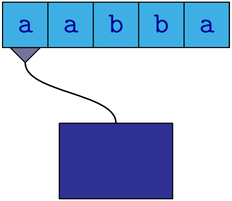
Examples of Nondeterministic Finite Automata

Example: An automaton recognizing the language over alphabet $\{\mathtt{a}, \mathtt{b}\}$ consisting of those words where the fifth symbol from the end is $\mathtt{a}$.

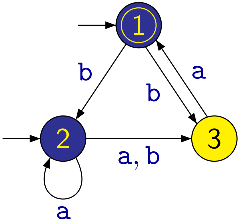


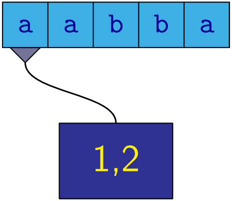
Transformation of NFA to DFA



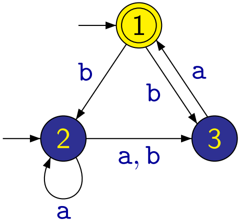


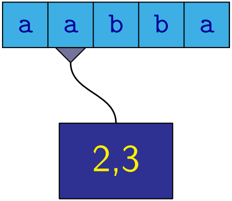
Transformation of NFA to DFA



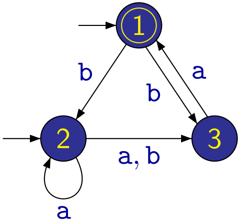


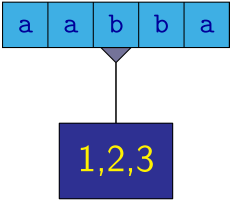
Transformation of NFA to DFA



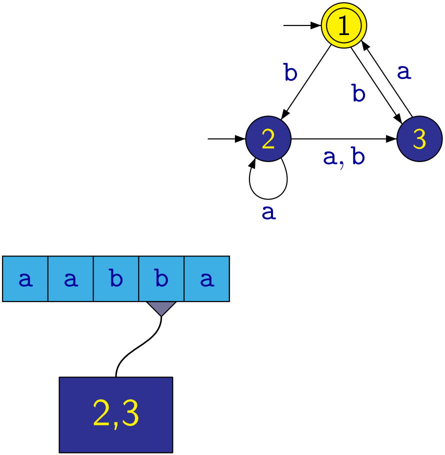


Transformation of NFA to DFA





Transformation of NFA to DFA



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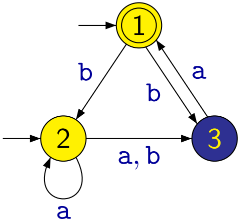

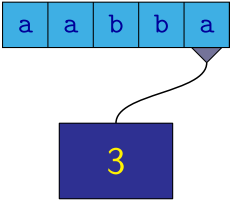

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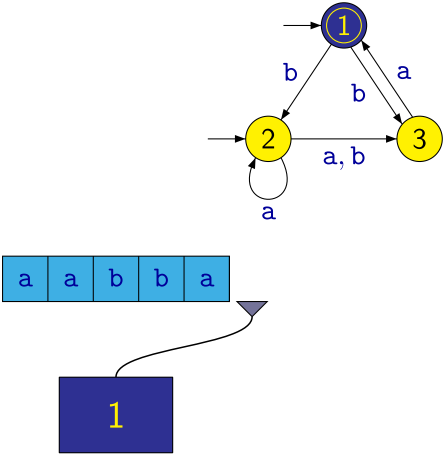

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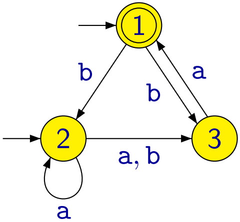

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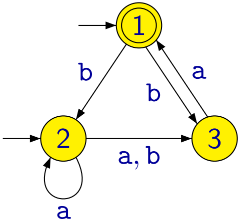

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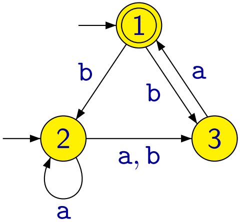

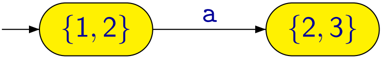

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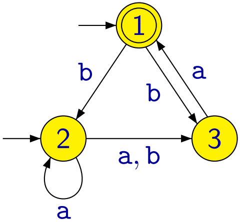

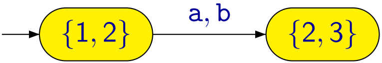

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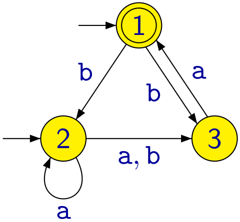

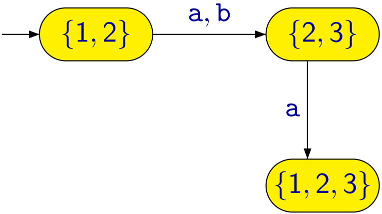

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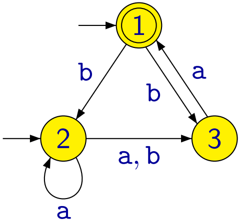

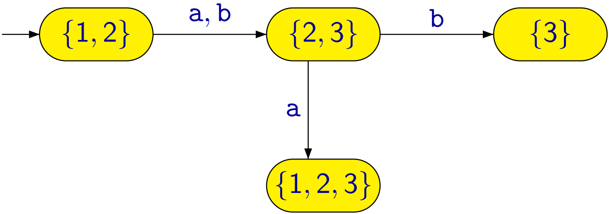

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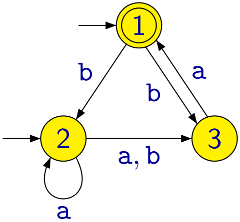

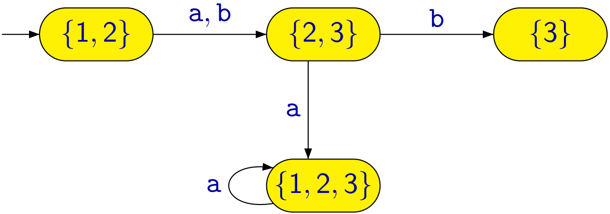

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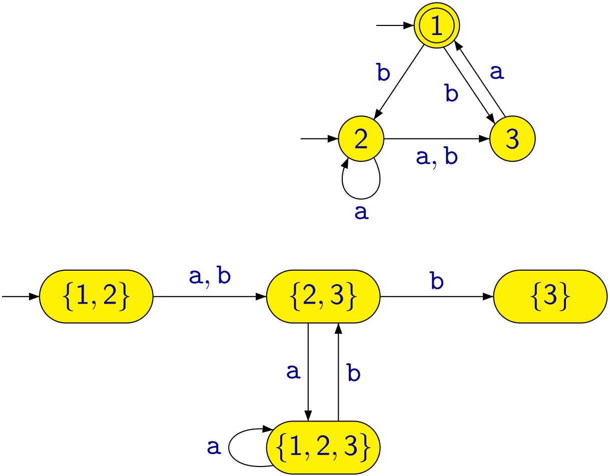

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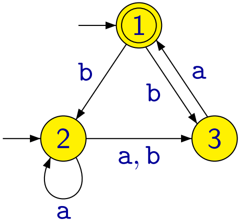

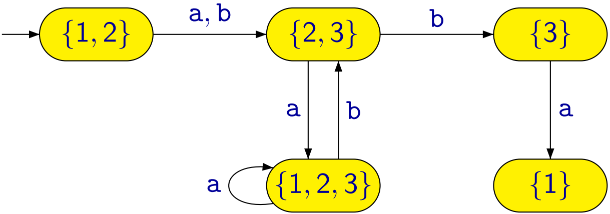

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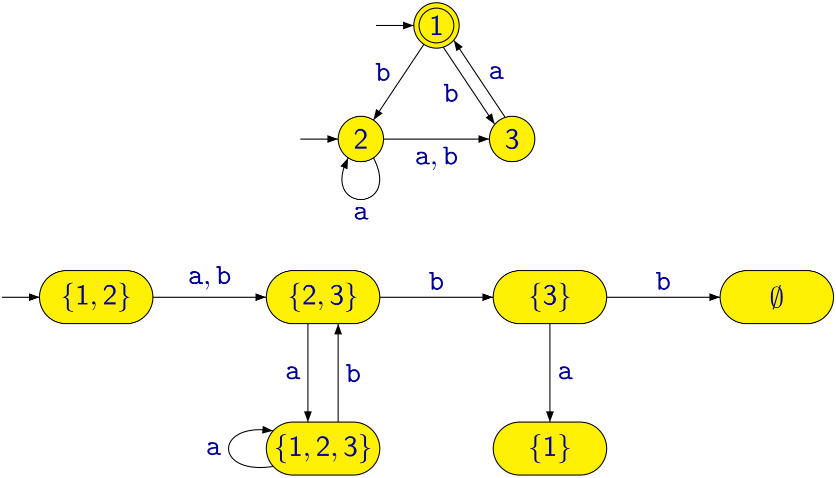

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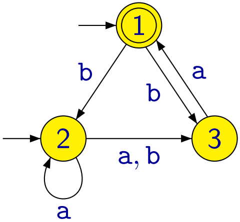

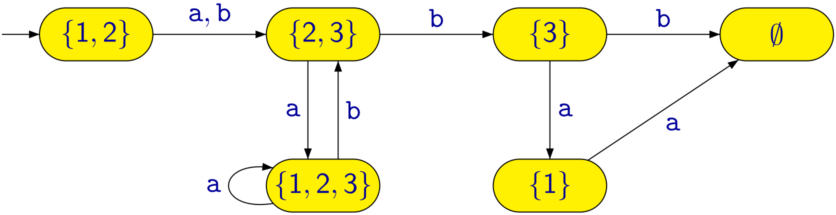

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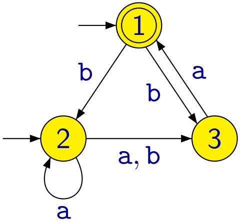

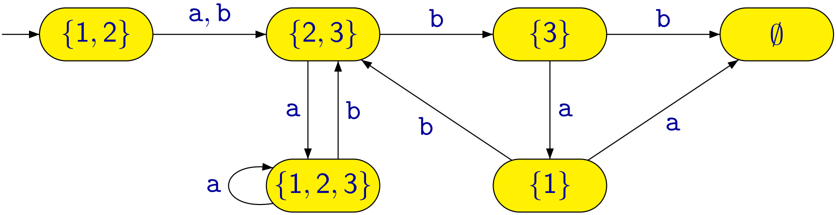

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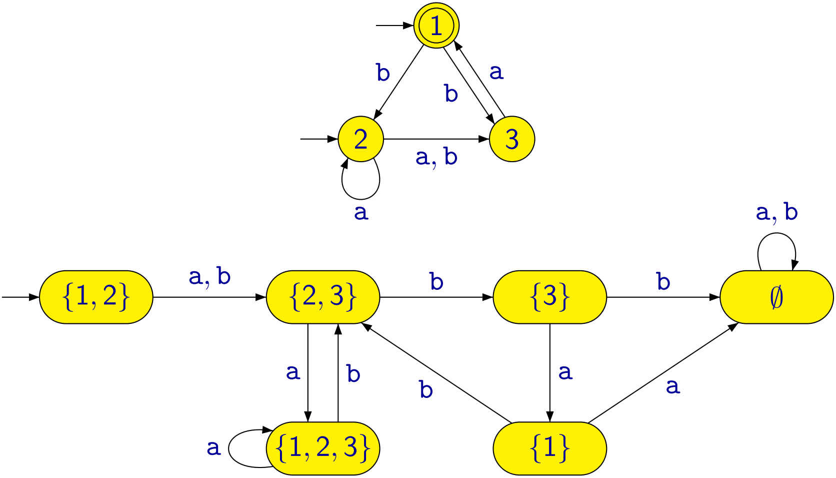

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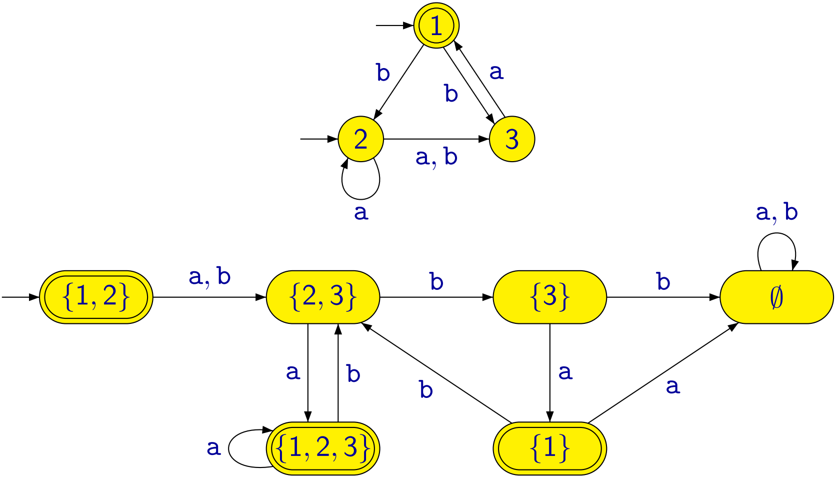


Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	
$\{2,3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	
$\leftarrow \{1, 2, 3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$		
$\{3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	
$\{3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\{3\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\{3\}$	$\{1\}$	

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	
$\leftarrow \{1\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$		
$\emptyset$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2,3$
$\rightarrow 2$	$2,3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\leftarrow \{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	
$\emptyset$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	–	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	–

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	$\{2, 3\}$
$\emptyset$		

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	–	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	–

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	$\{2, 3\}$
$\emptyset$	$\emptyset$	$\emptyset$

Transformation of NFA to DFA

	a	b
$\leftrightarrow 1$	$-$	$2, 3$
$\rightarrow 2$	$2, 3$	3
3	1	$-$

	a	b
$\leftrightarrow \{1, 2\}$	$\{2, 3\}$	$\{2, 3\}$
$\{2, 3\}$	$\{1, 2, 3\}$	$\{3\}$
$\leftarrow \{1, 2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$
$\{3\}$	$\{1\}$	$\emptyset$
$\leftarrow \{1\}$	$\emptyset$	$\{2, 3\}$
$\emptyset$	$\emptyset$	$\emptyset$

	a	b
$\leftrightarrow 1$	2	2
2	3	4
$\leftarrow 3$	3	2
4	5	6
$\leftarrow 5$	6	2
6	6	6

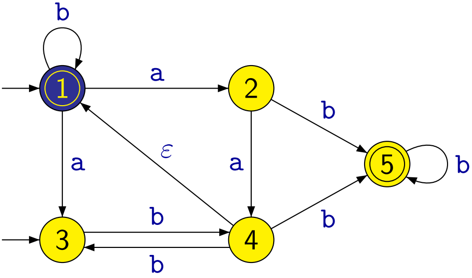
Transformation of NFA to DFA

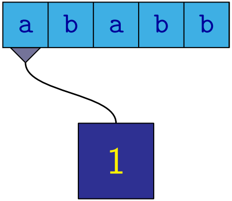
Remark: When a nondeterministic automaton with n states is transformed into a deterministic one, the resulting automaton can have 2^n states.

For example when we transform an automaton with 20 states, the resulting automaton can have $2^{20} = 1048576$ states.

It is often the case that the resulting automaton has far less than 2^n states. However, the worst cases are possible.

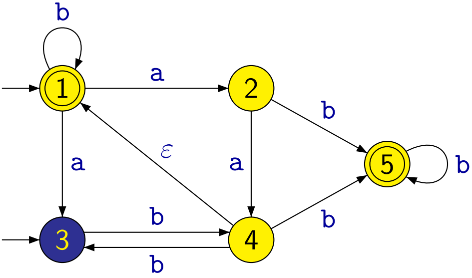
Generalized Nondeterministic Finite Automaton

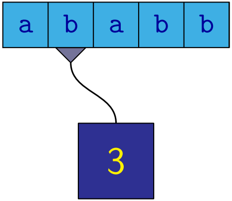




1

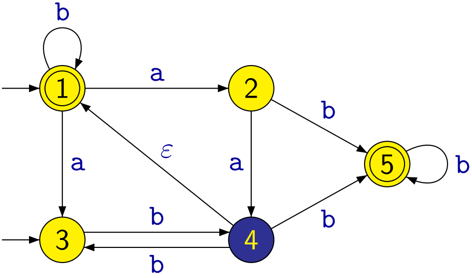
Generalized Nondeterministic Finite Automaton

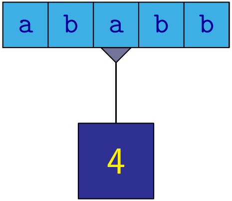




$1 \xrightarrow{\texttt{a}} 3$

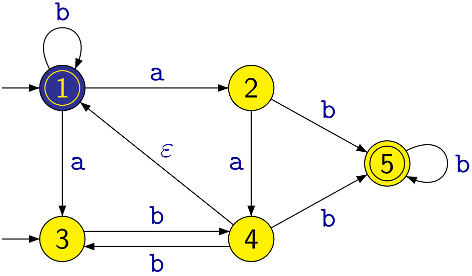
Generalized Nondeterministic Finite Automaton

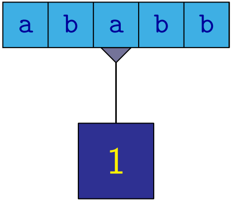




$$1 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 4$$

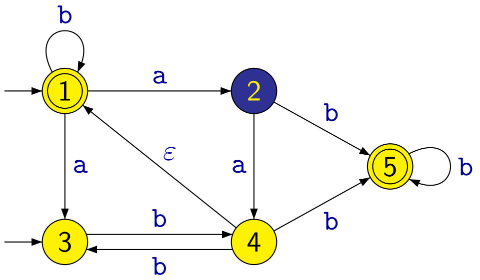
Generalized Nondeterministic Finite Automaton

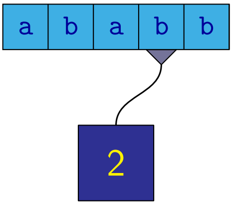




$$1 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 4 \xrightarrow{\varepsilon} 1$$

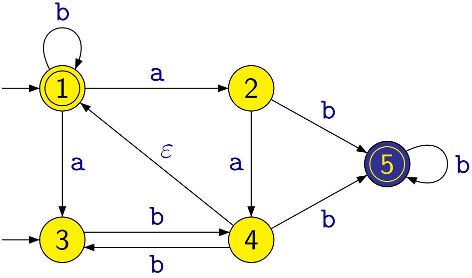
Generalized Nondeterministic Finite Automaton

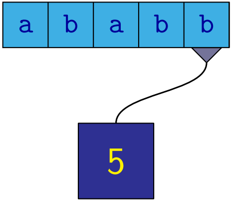




$$1 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 4 \xrightarrow{\varepsilon} 1 \xrightarrow{\texttt{a}} 2$$

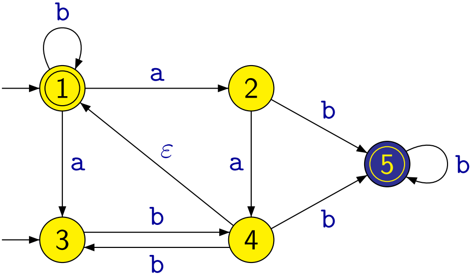
Generalized Nondeterministic Finite Automaton

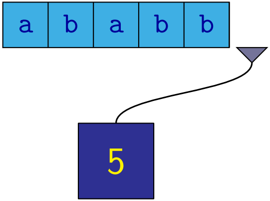




$$1 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 4 \xrightarrow{\varepsilon} 1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{b}} 5$$

Generalized Nondeterministic Finite Automaton





$1 \xrightarrow{\texttt{a}} 3 \xrightarrow{\texttt{b}} 4 \xrightarrow{\varepsilon} 1 \xrightarrow{\texttt{a}} 2 \xrightarrow{\texttt{b}} 5 \xrightarrow{\texttt{b}} 5$

Generalized Nondeterministic Finite Automaton

Compared to a nondeterministic finite automaton, a **generalized nondeterministic finite automaton** has the so called **ε-transitions**, i.e., transitions labelled with symbol ε.

When ε-transition is performed, only the state of the control unit is changed but the head on the tape is not moved.

Remark: The computations of a generalized nondeterministic automaton can be of an arbitrary length, even infinite (if the graph of the automaton contains a cycle consisting only of ε-transitions) regardless of the length of the word on the tape.

Generalized Nondeterministic Finite Automaton

Formally, a **generalized nondeterministic finite automaton** (**GNFA**) is defined as a tuple

$$(Q, \Sigma, \delta, I, F)$$

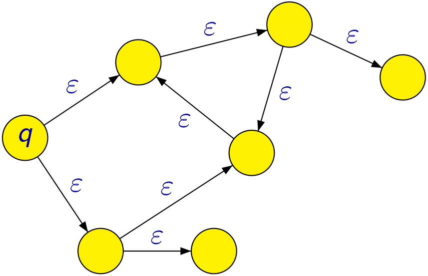
where:

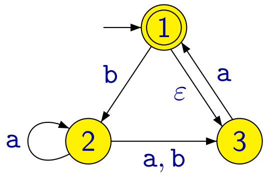
- Q is a finite set of **states**
- Σ is a finite **alphabet**
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$ is a **transition function**
- $I \subseteq Q$ is a set of **initial states**
- $F \subseteq Q$ is a set of **accepting states**

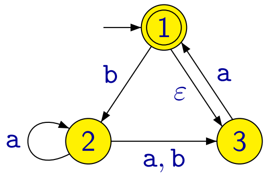
Remark: NFA can be viewed as a special case of GNFA, where $\delta(q, \varepsilon) = \emptyset$ for all $q \in Q$.

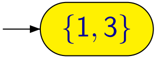
Transformation to a Deterministic Finite Automaton

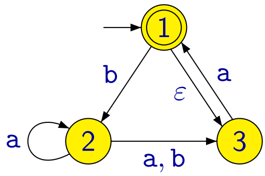
A generalized nondeterministic finite automaton can be transformed into a deterministic one using a similar construction as a nondeterministic finite automaton with the difference that we add to sets of states also all states that are reachable from already added states by some sequence of ε-transitions.

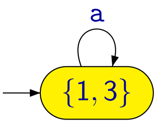


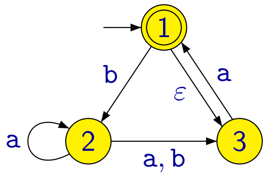

1
2
3
a
b
a
ε
a, b

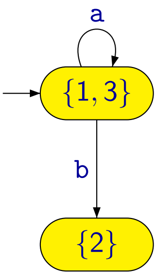

1
b
a
ε
a
2
3
a, b

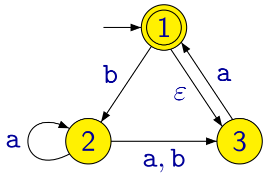

{1, 3}


1
b
a
ε
a
2
3
a, b

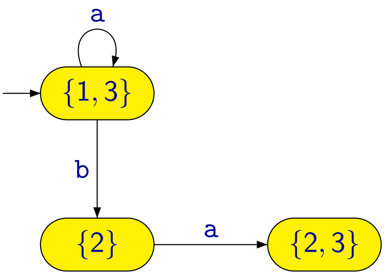

a
{1, 3}


1
b
a
ε
a
2
a, b
3

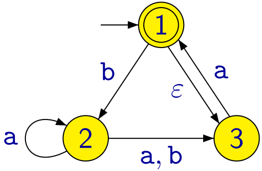

a
{1, 3}
b
{2}

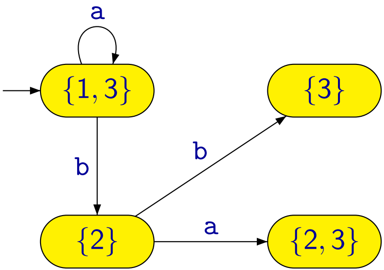


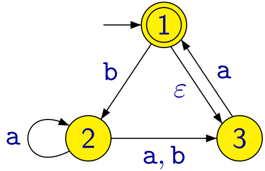
1
b
a
ε
2
a
3
a, b

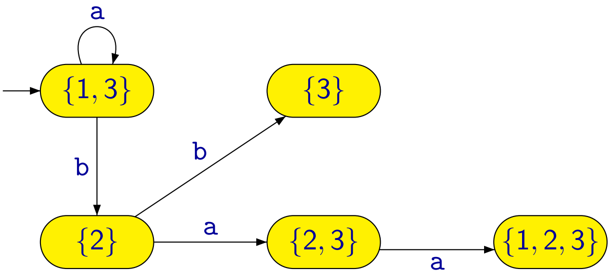


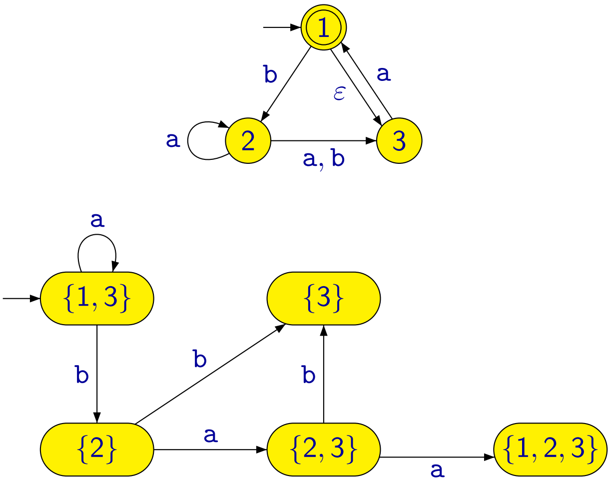
a
{1, 3}
b
{2}
a
{2, 3}

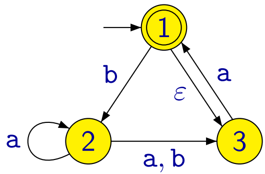

1
b
a
ε
a
2
3
a, b

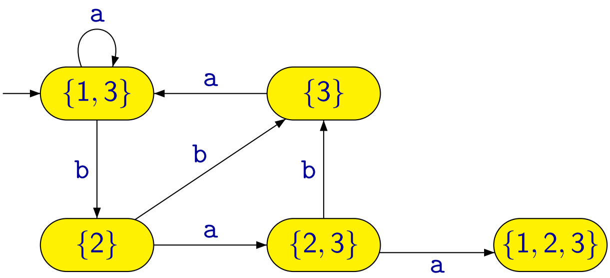

a
{1, 3}
{3}
b
b
{2}
a
{2, 3}

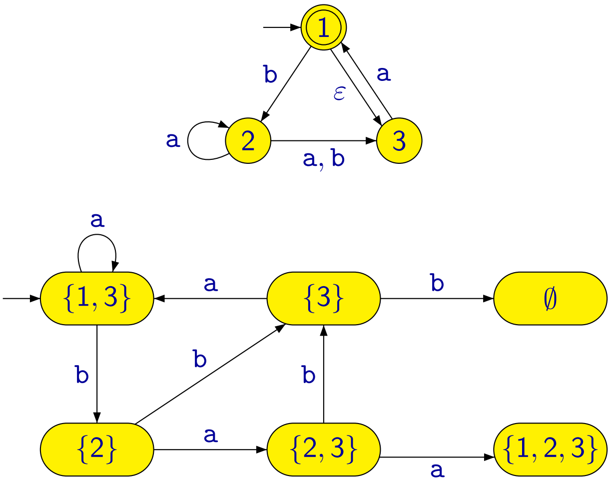

1
b
ε
a
a
2
3
a, b

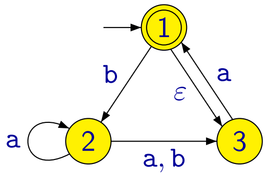

a
{1, 3}
{3}
b
b
{2}
a
{2, 3}
a
{1, 2, 3}

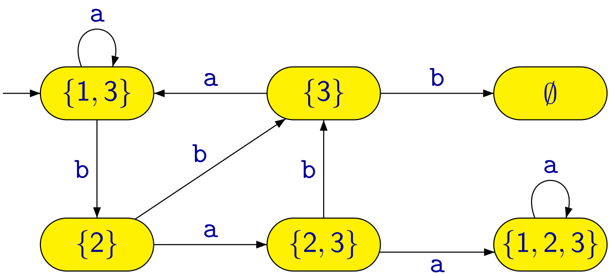

1
b
a
ε
a
2
a, b
3
a
{1, 3}
{3}
b
b
b
{2}
a
{2, 3}
a
{1, 2, 3}

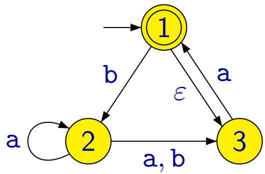

1
2
3
b
a
ε
a
a, b

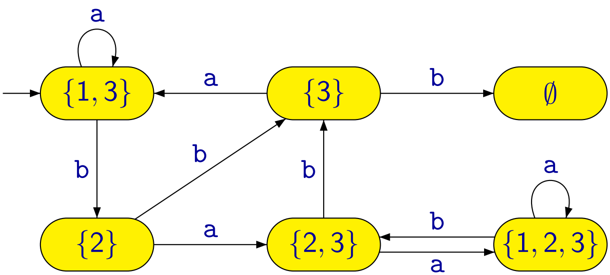

a
{1, 3}
a
{3}
b
b
b
{2}
a
{2, 3}
a
{1, 2, 3}

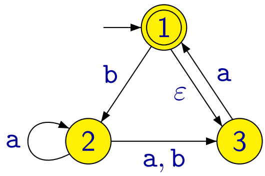

1
b
a
ε
a
2
3
a, b
a
a
b
{1, 3}
{3}
∅
b
b
b
{2}
a
{2, 3}
a
{1, 2, 3}

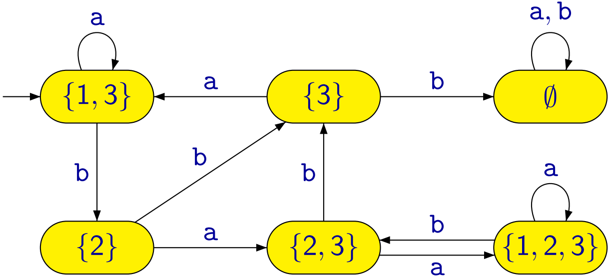

1
b
ε
a
a
2
3
a, b

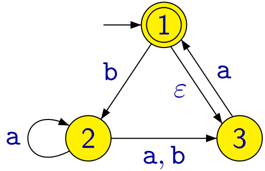

a
{1, 3}
a
{3}
b
∅
b
b
b
a
{2}
a
{2, 3}
a
{1, 2, 3}

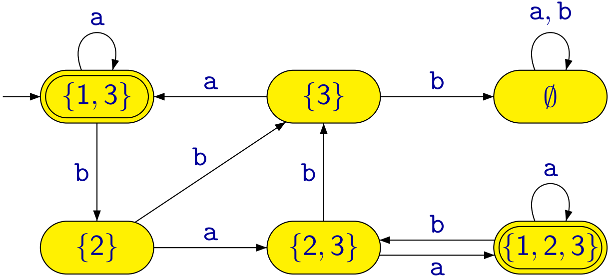

1
b
a
ε
a
2
3
a, b


a
{1, 3}
a
{3}
b
∅
b
b
b
a
{2}
a
{2, 3}
b
a
{1, 2, 3}


1
b
ε
a
a
2
3
a, b


a
a, b
a
b
{1, 3}
{3}
∅
b
b
b
a
{2}
a
{2, 3}
b
{1, 2, 3}
a


1
b
ε
a
a
2
3
a, b


a
a, b
{1, 3}
a
{3}
b
∅
b
b
b
a
{2}
a
{2, 3}
b
{1, 2, 3}
a

Transformation of GNFA to DFA

Before formally describing the transition of GNFA to DFA, let us introduce some auxiliary definitions.

Let us assume some given GNFA $\mathcal{A} = (Q, \Sigma, \delta, I, F)$.

Let us define the function $\hat{\delta} : \mathcal{P}(Q) \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$ so that for $K \subseteq Q$ and $a \in \Sigma \cup \{\varepsilon\}$ there is

$$\hat{\delta}(K, a) = \bigcup_{q \in K} \delta(q, a)$$

Transformation of GNFA to DFA

For $K \subseteq Q$, let $Cl_\varepsilon(K)$ be all the states reachable from the states from the set K by some arbitrary sequence of ε-transitions.

This means that the function $Cl_\varepsilon : \mathcal{P}(Q) \to \mathcal{P}(Q)$ is defined so that for $K \subseteq Q$ is $Cl_\varepsilon(K)$ the smallest (with respect to inclusion) set satisfying the following two conditions:

- $K \subseteq Cl_\varepsilon(K)$
- For each $q \in Cl_\varepsilon(K)$ it holds that $\delta(q, \varepsilon) \subseteq Cl_\varepsilon(K)$.

Remark: Let us note that $Cl_\varepsilon(Cl_\varepsilon(K)) = Cl_\varepsilon(K)$ for arbitrary K.

Let us also note that in the case of NFA (where $\delta(q, \varepsilon) = \emptyset$ for each $q \in Q$) is $Cl_\varepsilon(K) = K$.

Transformation of GNFA to DFA

For a given GNFA $\mathcal{A} = (Q, \Sigma, \delta, I, F)$ we can now construct DFA $\mathcal{A}' = (Q', \Sigma, \delta', q'_0, F')$, where:

- $Q' = \mathcal{P}(Q)$ (so $K \in Q'$ means that $K \subseteq Q$)
- $\delta' : Q' \times \Sigma \to Q'$ is defined so that for $K \in Q'$ and $a \in \Sigma$:

$$\delta'(K, a) = Cl_\varepsilon(\hat{\delta}(Cl_\varepsilon(K), a))$$

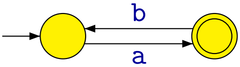
- $q'_0 = Cl_\varepsilon(I)$
- $F' = \{K \in Q' \mid Cl_\varepsilon(K) \cap F \neq \emptyset\}$

It is not difficult to verify that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

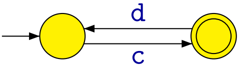
Concatenation of Languages

$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

$\mathcal{A}_1$:



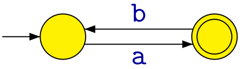
$\mathcal{A}_2$:



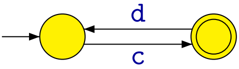
Concatenation of Languages

$\Sigma = \{a, b, c, d\}$

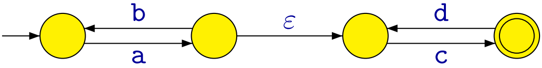
$\mathcal{A}_1$:



$\mathcal{A}_2$:



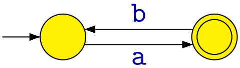
$\mathcal{A}$:



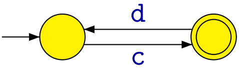
$\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}(\mathcal{A}_2)$

$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

$\mathcal{A}_1$:

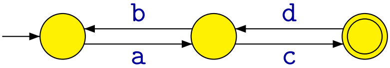


$\mathcal{A}_2$:



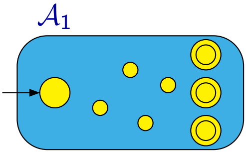
An incorrect construction:

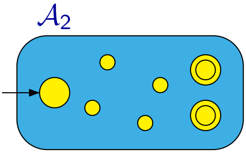
$\mathcal{A}$:



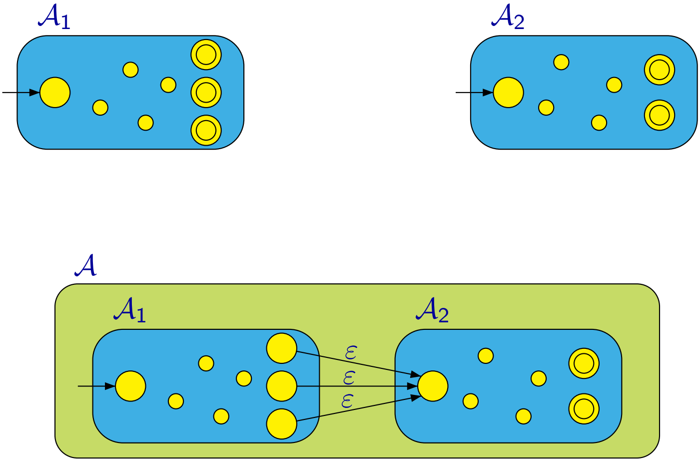
$\mathtt{acdbac} \in \mathcal{L}(\mathcal{A})$ but $\mathtt{acdbac} \notin \mathcal{L}(\mathcal{A}_1) \cdot \mathcal{L}(\mathcal{A}_2)$

Concatenation of Languages

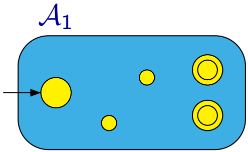




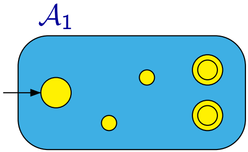
Concatenation of Languages

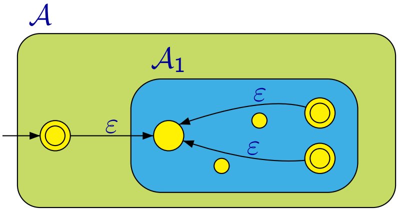


Iteration of a Language



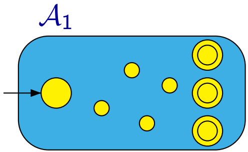
Iteration of a Language

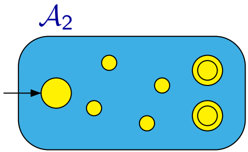




Union of Languages

An alternative construction for the union of languages:





Union of Languages

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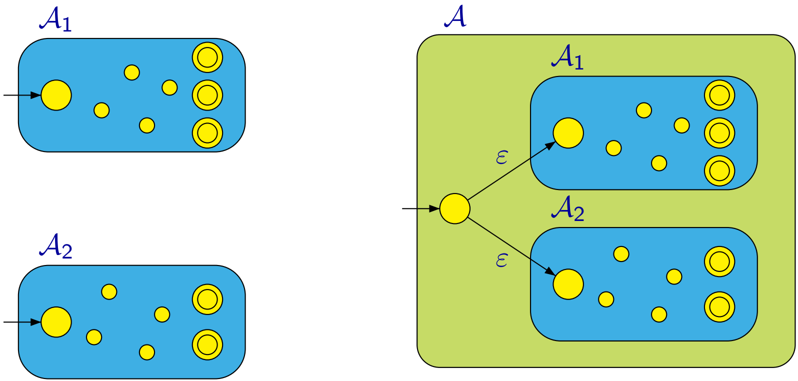

Closure Properties of the Class of Regular Languages

The set of (all) regular languages is closed with respect to:

- union
- intersection
- complement
- concatenation
- iteration
- ...