Theoretical Computer Science

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Lecturer

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On these pages you will find:

- Information about the course
- Slides from lectures
- Exercises for tutorials
- Recent news for the course
- A link to a page with animations

Requirements

- **Credit** (38 points):
  - **Presentation** (10 points) — it is necessary to obtain at least 5 points;
    a correction is possible for at most 5 points, for the correction, at least 1 point must be obtained
  - **Written test** (21 points) — it is necessary to obtain at least 7 points
  - **Activity on exercises** (7 points)
- **Exam** (62 points)
  - A written exam that constists of two parts, each for 31 points; it is necessary to obtain at least 11 points from each part, and at least 25 points in total.

Theoretical Computer Science

Theoretical computer science — a scientific field on the border between computer science and mathematics

- investigation of general questions concerning algorithms and computations
- study of different kinds of formalisms for description of algorithms
- study of different approaches for description of syntax and semantics of formal languages (mainly programming languages)
- a mathematical approach to analysis and solution of problems (proofs of general mathematical propositions concerning algorithms)

Theoretical Computer Science

Examples of some typical questions studied in theoretical computer science:

- Is it possible to solve the given problem using some algorithm?
- If the given problem can be solved by an algorithm, what is the computational complexity of this algorithm?
- Is there an efficient algorithm solving the given problem?
- How to check that a given algorithm is really a correct solution of the given problem?
- What kinds instructions are sufficient for a given machine to perform a given algorithm?

Algorithms and Problems

Algorithm — mechanical procedure that computes something (it can be executed by a computer)

Algorithms are used for solving **problems**.

An example of an algorithmic problem:

Input:	Natural numbers x and y.
Output:	Natural number z such that $z = x + y$.

Algorithms and Problems

Algorithm — mechanical procedure that computes something (it can be executed by a computer)

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An example of an algorithmic problem:

Input: Natural numbers x and y.

Output: Natural number z such that $z = x + y$.

A particular input of a problem is called an **instance** of the problem.

Example: An example of an instance of the problem given above is a pair of numbers 728 and 34.

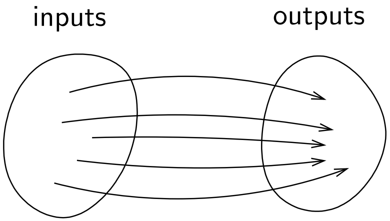
The corresponding output for this instance is number 762.

Problems

Problem

When specifying a **problem** we must determine:

- what is the set of possible inputs
- what is the set of possible outputs
- what is the relationship between inputs and outputs



Examples of Problems

Problem "Sorting"

Input: A sequence of elements $a_1, a_2, \ldots, a_n$.

Output: Elements of the sequence $a_1, a_2, \ldots, a_n$ ordered from the least to the greatest.

Example:

- Input: $8, 13, 3, 10, 1, 4$
- Output: $1, 3, 4, 8, 10, 13$

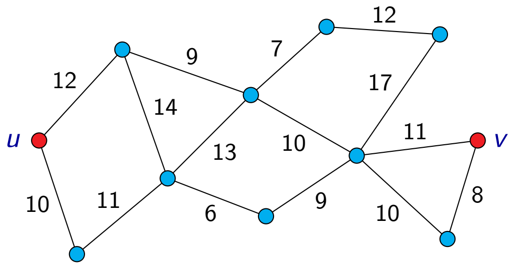
An example of an algorithmic problem

Problem "Finding the shortest path in an (undirected) graph"

Input: An undirected graph $G = (V, E)$ with edges labelled with numbers, and a pair of nodes $u, v \in V$.

Output: The shortest path from node u to node v.

Example:



Algorithms and Problems

An algorithm **solves** a given problem if:

- For each input, the computation of the algorithm halts after a finite number of steps.
- For each input, the algorithm produces a correct output.

Correctness of an algorithm — verifying that the algorithm really solves the given problem

Computational complexity of an algorithm:

- **time complexity** — how the running time of the algorithm depends on the size of input data
- **space complexity** — how the amount of memory used by the algorithm depends on the size of input data

Remark: For one problem there can be many diffent algorithms that correctly solve the problem.

Other Examples of Problems

Problem "Primality"

Input: A natural number n.

Output: YES if n is a prime, NO otherwise.

Remark: A natural number n is a **prime** if it is greater than 1 and is divisible only by numbers 1 and n.

Few of the first primes: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$

Decision Problems

The problems, where the set of outputs is $\{\textsc{Yes}, \textsc{No}\}$ are called **decision problems**.

Decision problems are usually specified in such a way that instead of describing what the output is, a question is formulated.

Example:

Problem "Primality"

Input: A natural number n.

Question: Is n a prime?

Optimization Problems

Those problems where for each input instance there is a corresponding set of **feasible solutions** and where the aim is to select between these feasible solutions that is some respect minimal or maximal (or possibly to find out that there are no feasible solutions), are called **optimization problems**.

Example:

Problem "Finding the shortest path in an (undirected) graph"

Input: An undirected graph $G = (V, E)$ with edges labelled with numbers, and a pair of nodes $u, v \in V$.

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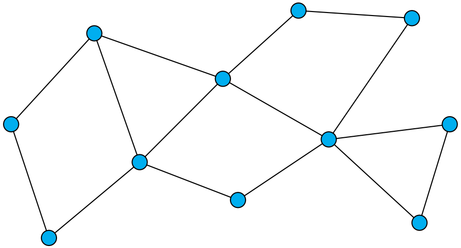
Optimization Problems

Problem "Coloring of a graph"

Input: An undirected graph G.

Output: The minimal number of colors to color the nodes of the graph G in such a way that no two nodes connected with an edge are colored with the same color, and a concrete example of such coloring using this minimal number of colors.



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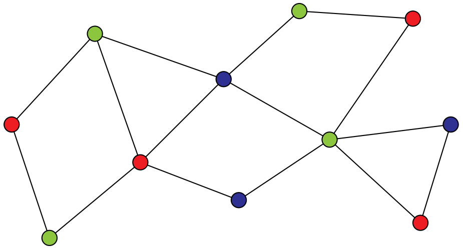


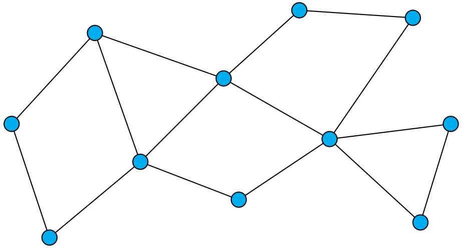
Colors: 3

Optimization Problems

Problem "Coloring of a graph with k colors"

Input: An undirected graph G and a natural number k.

Question: Is it possible to color the nodes of the graph G with k colors in such a way that no two nodes connected with an edge are colored with the same color?



$k = 3$

Optimization Problems

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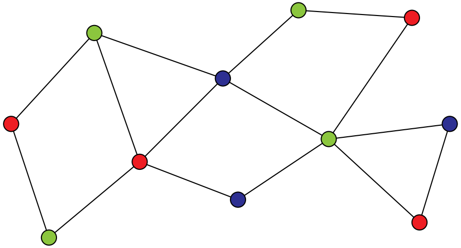

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Algorithmically Solvable Problems

Let us assume we have a problem P.

If there is an algorithm solving the problem P then we say that the problem P is **algorithmically solvable**.

If P is a decision problem and there is an algorithm solving the problem P then we say that the problem P is **decidable (by an algorithm)**.

If we want to show that a problem P is algorithmically solvable, it is sufficient to show some algorithm solving it (and possibly show that the algorithm really solves the problem P).

Algorithmically Unsolvable Problems

A problem that is not algorithmically solvable is **algorithmically unsolvable**.

A decision problem that is not decidable is **undecidable**.

Surprisingly, there are many (exactly defined) problems, for which it was proved that they are not algorithmically solvable.

Computability theory — area of theoretical computer science studying, which problems can be solved algorithmically and which cannot.

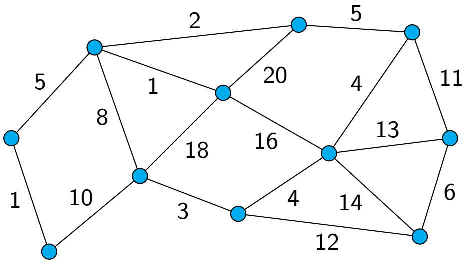
Complexity Theory

Many problems are algorithmically solvable but there do not exist (or are not known) efficient algorithms solving them:

TSP - traveling salesman problem

Input: An undirected graph G with edges labelled with natural numbers.

Output: A shortest closed path that goes through all vertices of the graph.



Theoretical Computer Science

Some other areas of theoretical computer science:

- complexity theory
- theory of formal languages
- models of computation
- parallel and distributed algorithms
- ...

Theory of Formal Languages

An area of theoretical computer science dealing with questions concerning **syntax**.

- **Language** — a set of words
- **Word** — a sequences of symbols from some alphabet
- **Alphabet** — a set of **symbols** (or **letters**)

Words and languages appear in computer science on many levels:

- Representation of input and output data
- Representation of programs
- Manipulation with character strings or files
- ...

Theory of Formal Languages – Motivation

Examples of problem types, where theory of formal languages is useful:

- Construction of compilers:
  - Lexical analysis
  - Syntactic analysis
- Searching in text:
  - Searching for a given text pattern
  - Seaching for a part of text specified by a regular expression

Alphabet, Word

- **Alphabet** — a nonempty finite set of **symbols**

  Example: $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$

- **Word** — a finite sequence of symbols from the given alphabet

  Example: `cabcbba`

  The set of all words of alphabet Σ is denoted with Σ^*.

  For variables, whose values are words, we will use names such as w, u, v, x, y, z, etc., possibly with indexes (e.g., w_1, w_2)

  So when we write $w = \mathtt{cabcbba}$, it means that the value of variable w is word `cabcbba`.

  Similarly, the notation $w \in \Sigma^*$ means that the value of a variable w is some word consisting of symbols belonging to alphabet Σ.

Formal Languages

Definition

A **(formal) language** L over an alphabet Σ is a subset of Σ^*, i.e., $L \subseteq \Sigma^*$.

Example: Let us assume that $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$:

- Language $L_1 = \{\, \mathtt{aab}, \mathtt{bcca}, \mathtt{aaaaa} \,\}$
- Language
  $L_2 = \{\, w \in \Sigma^* \mid \text{the number of occurrences of } \mathtt{b} \text{ in } w \text{ is even} \,\}$

Formal Languages

Example:

Alphabet Σ is the set of all ASCII characters.

Example of a word:

```
#include <stdio.h>

int main()
{
    printf("Hello, world!\n");
    return 0;
}
```

`#include␣<stdio.h>` $\hookleftarrow\hookleftarrow$ `int␣main()` $\hookleftarrow$ `{` $\hookleftarrow$ `␣␣␣␣printf("He` $\cdots$

Formal Languages

Formalisms used for description of formal languages:

- automata
- grammars
- regular expressions

Encoding of Input and Output

Inputs and outputs of an algorithm could be encoded as words over some alphabet Σ.

Example: For example, for problem "Sorting" we can take alphabet $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ,\}$.

An example of input data (as a word over alphabet Σ):

```
826,13,3901,128,562
```

and the corresponding output data (as a word over alphabet Σ)

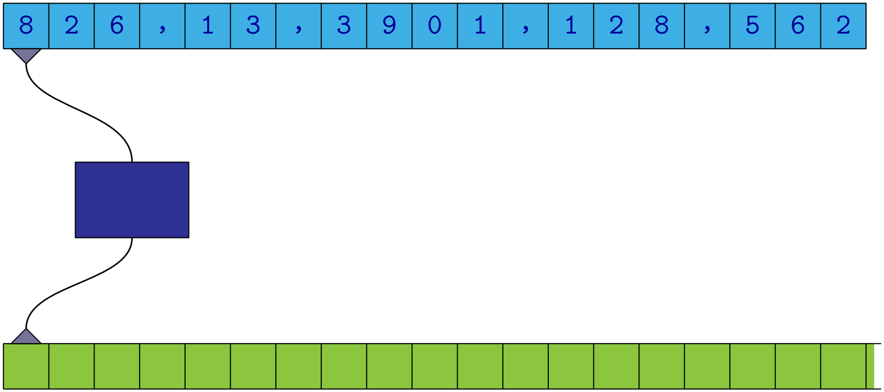
```
13,128,562,826,3901
```

Remark: It is often the case that only some words over the given alphabet represent valid input or output.

Input/Output Behaviour of an Algorithm

We can assume that the algorithm is executed on a certain type of machine.

Input



Output

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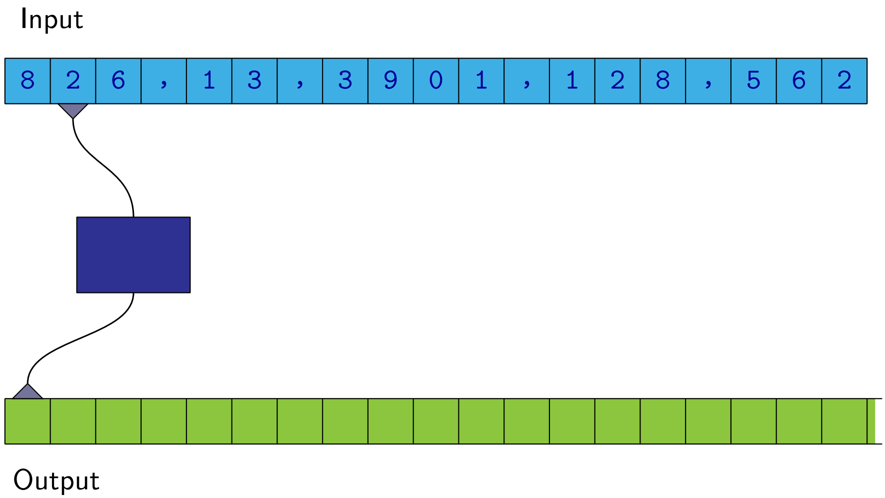

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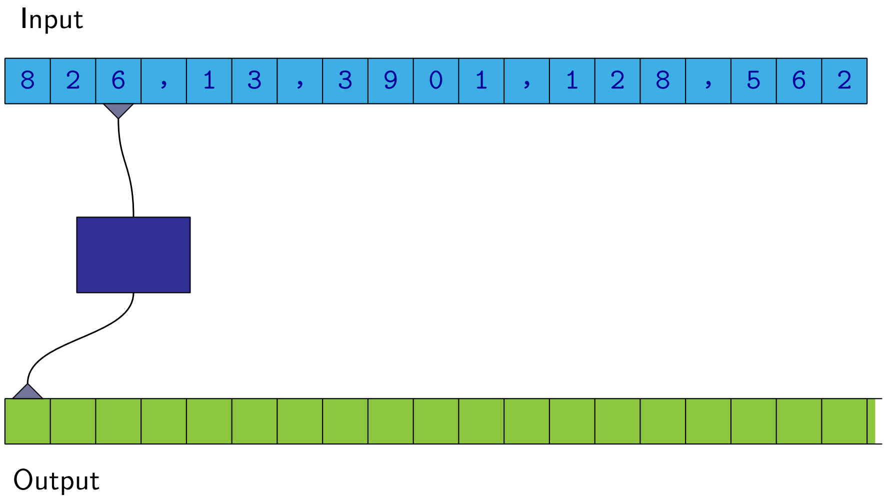

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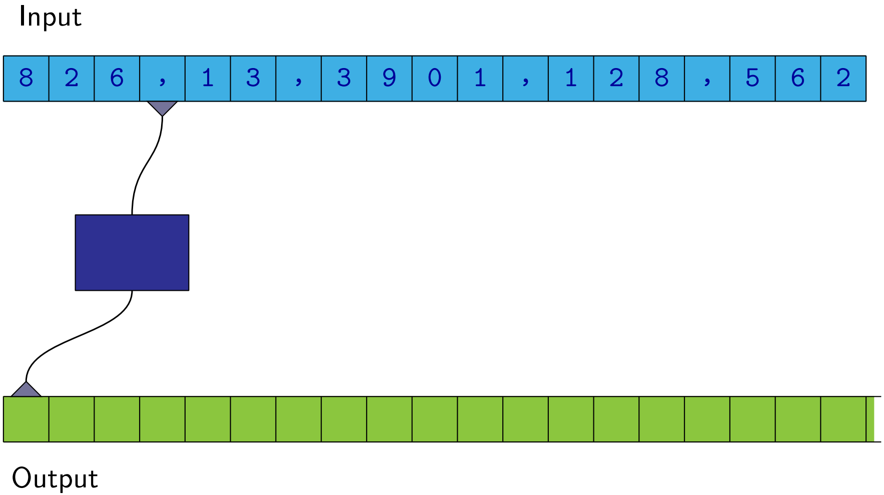

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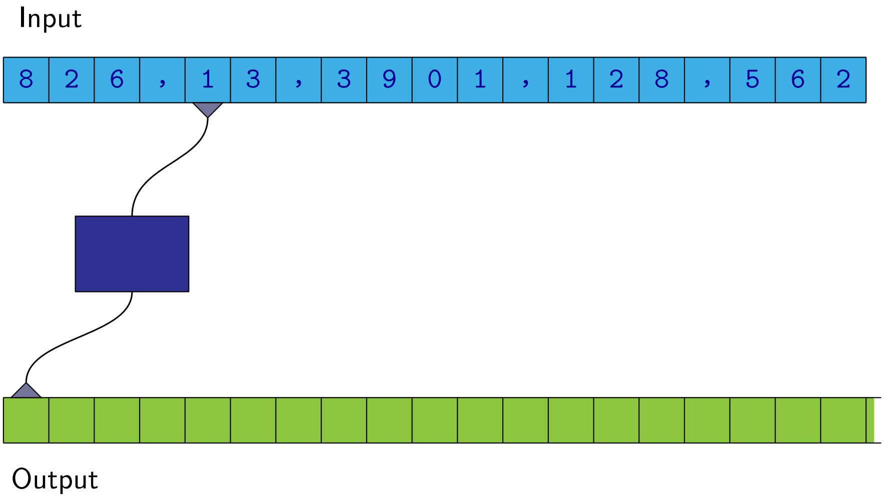

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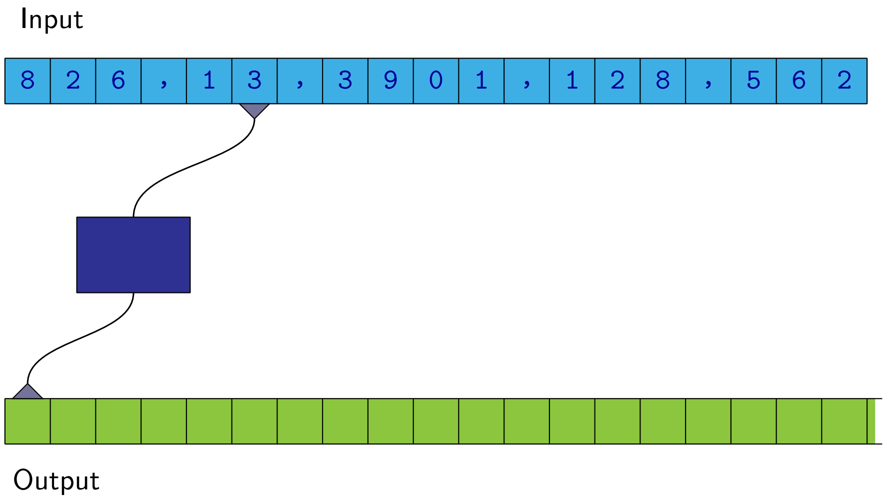

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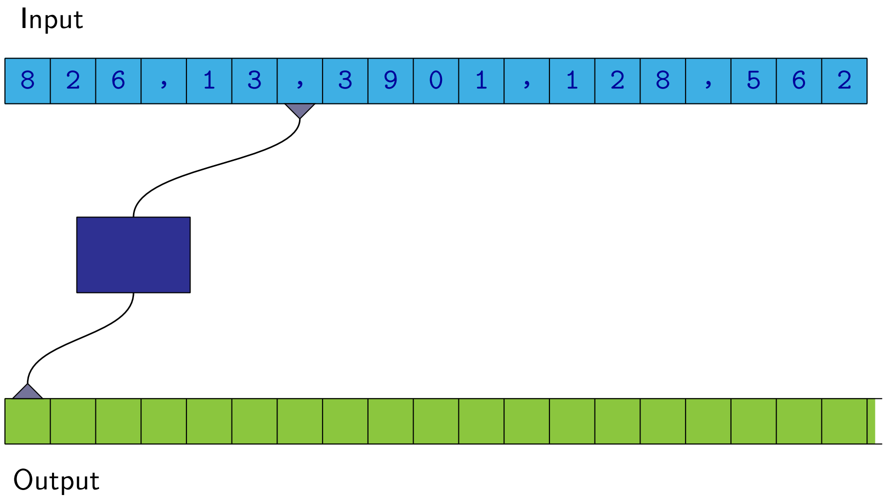

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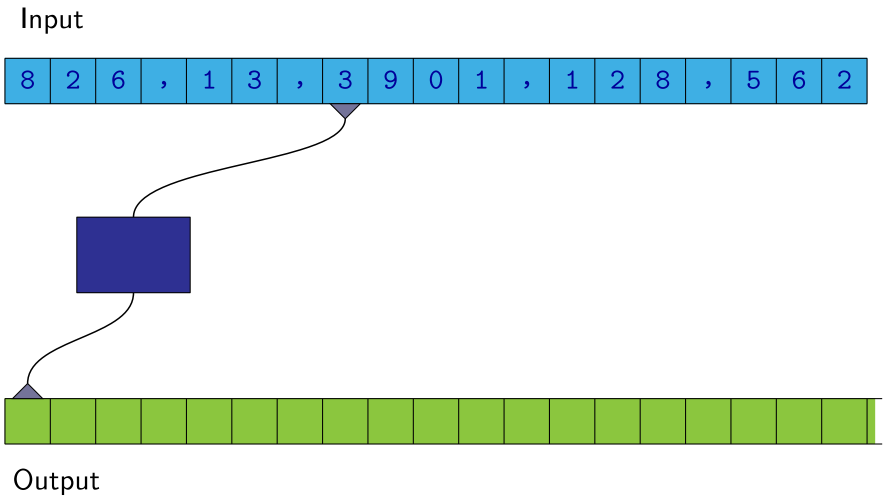

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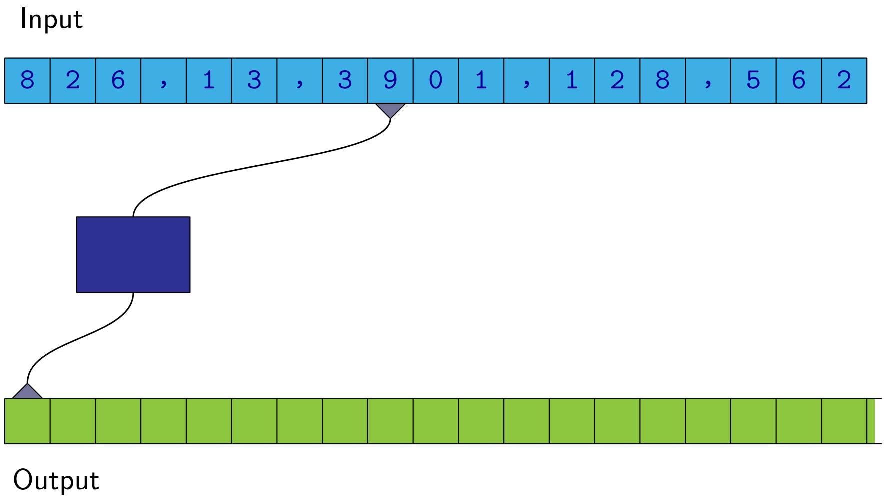

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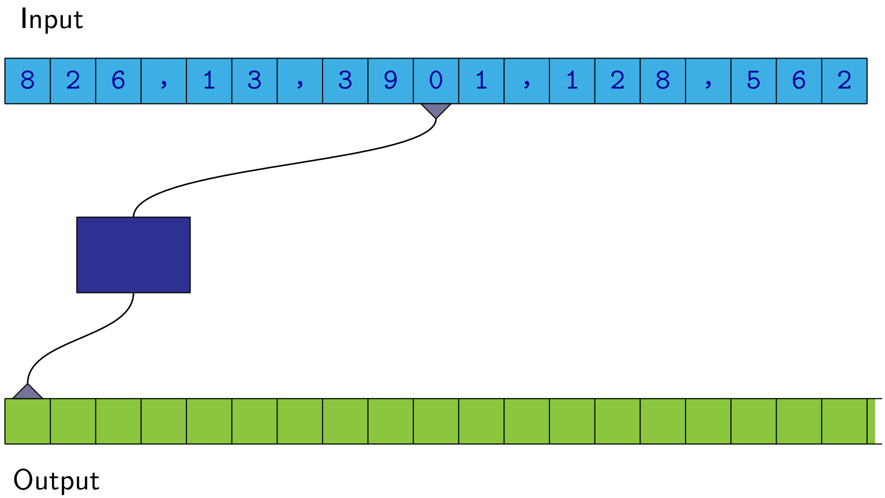

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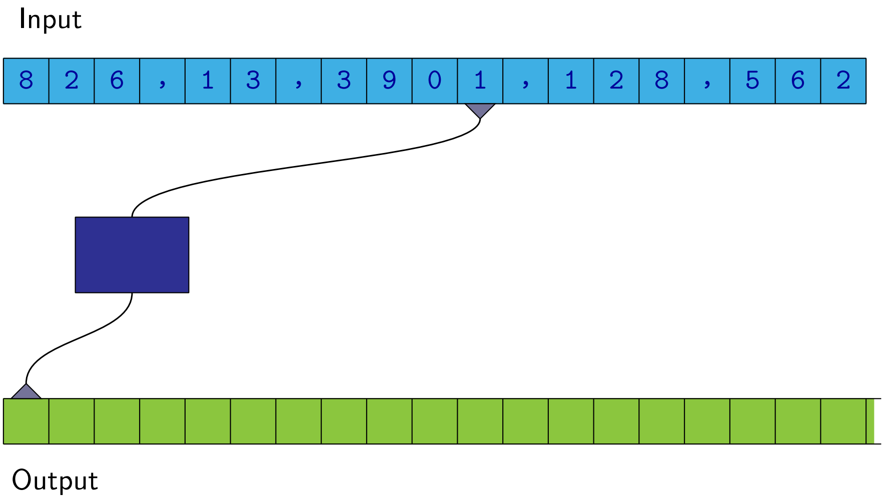

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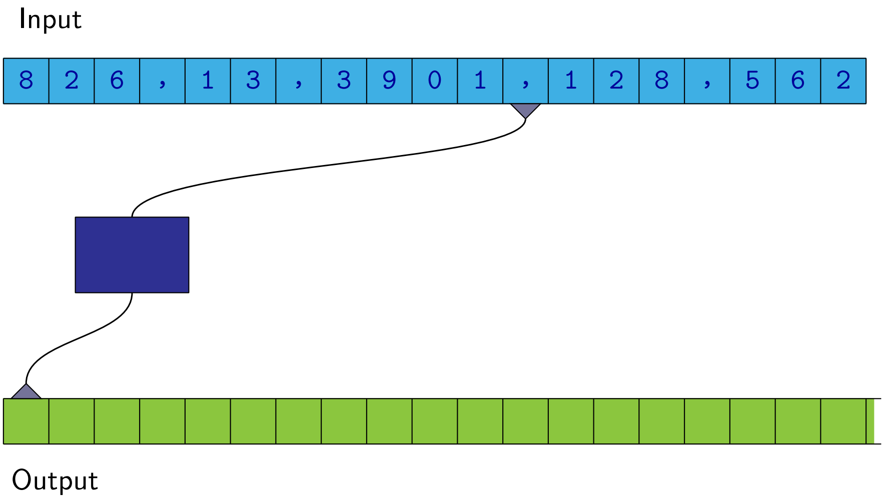

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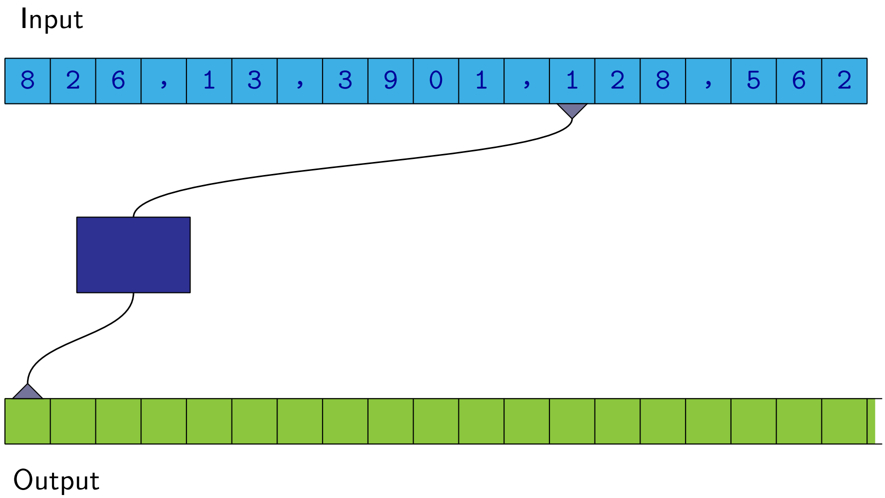

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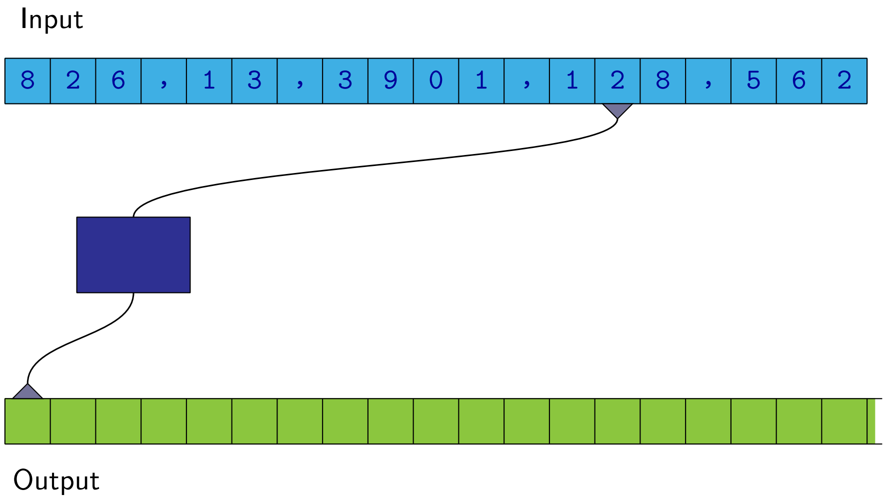

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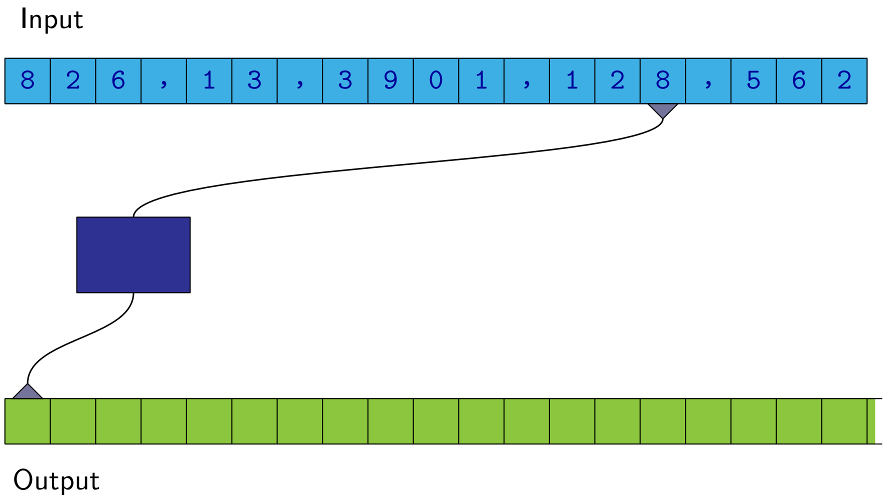

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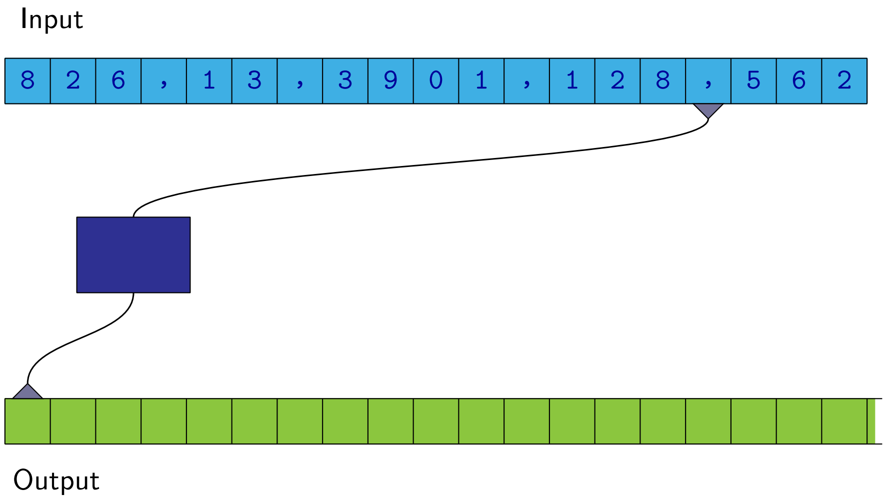

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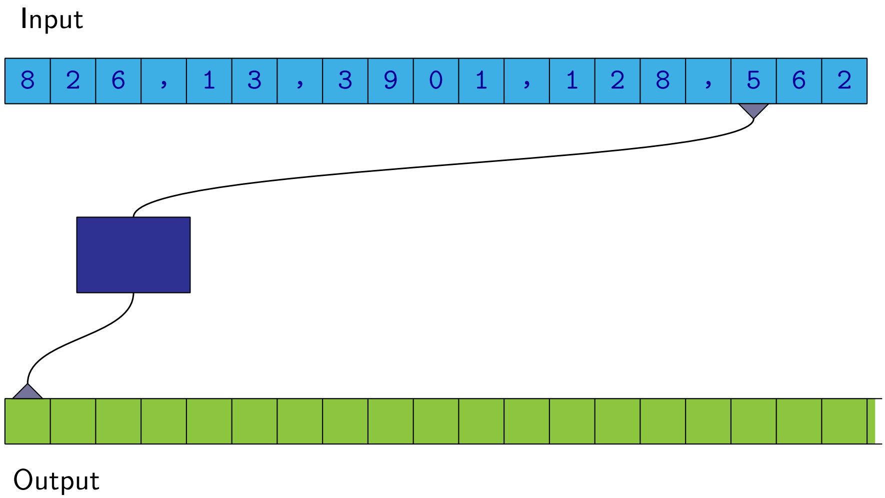

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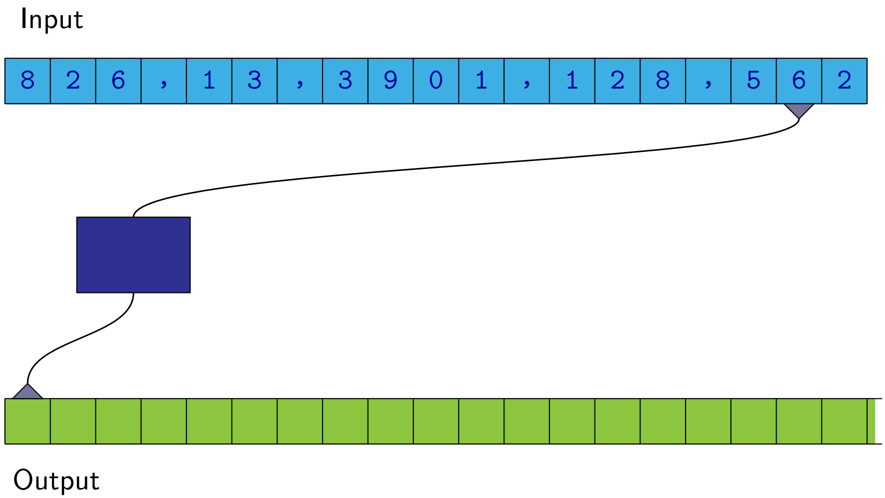

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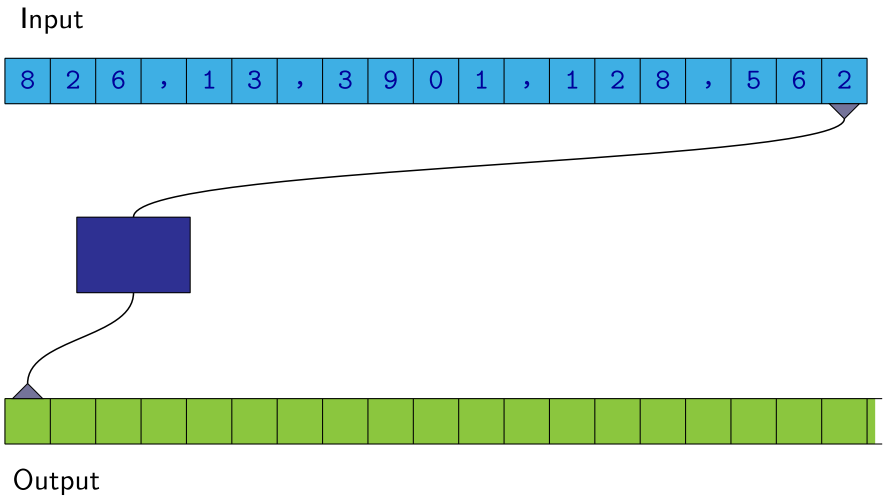

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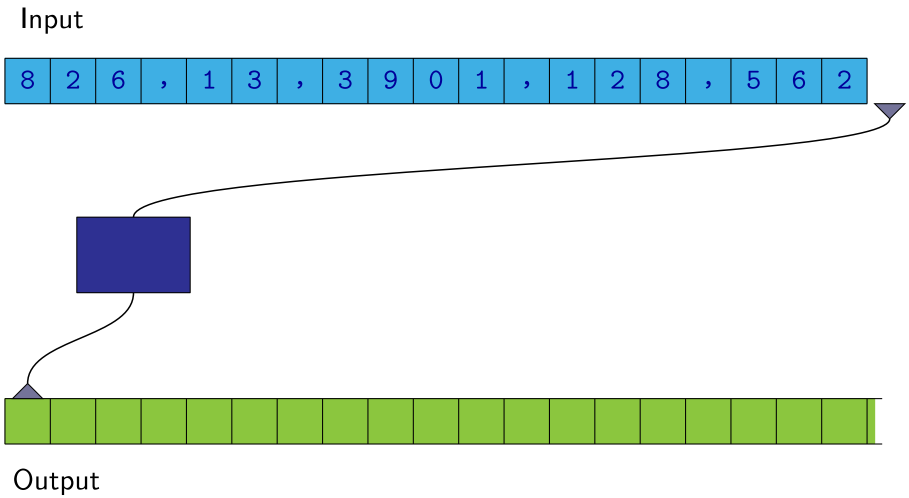

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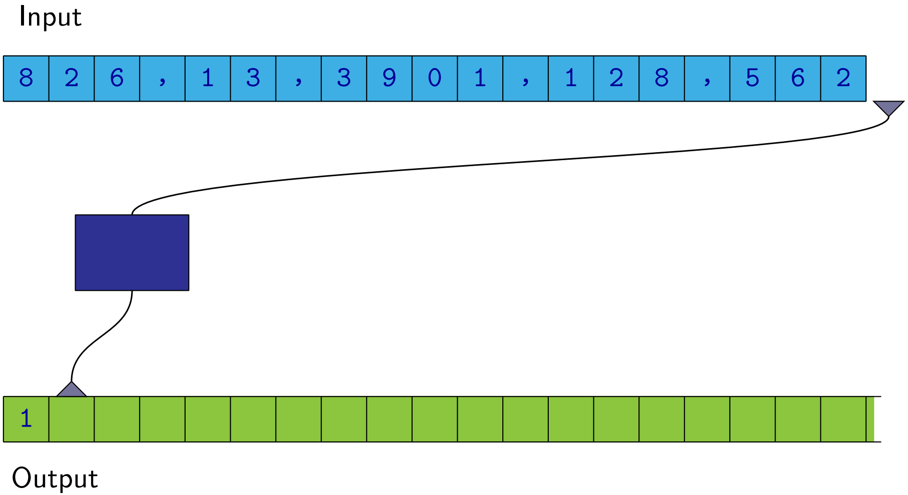

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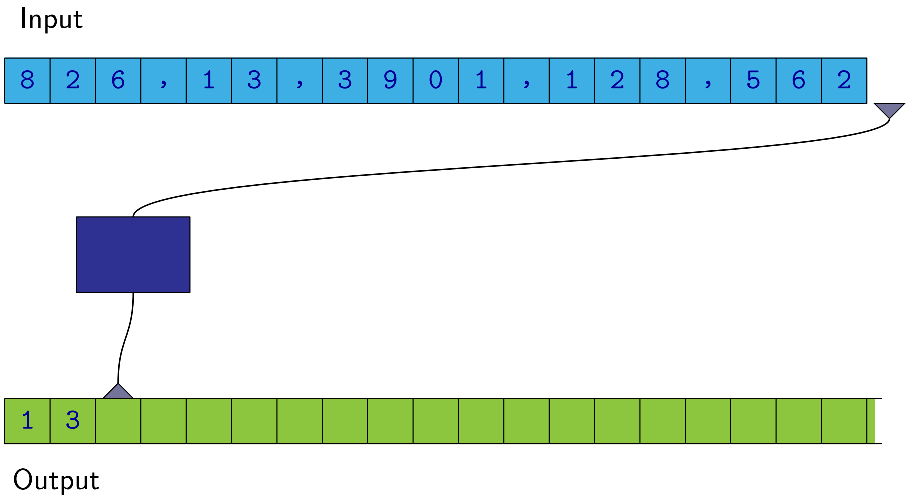

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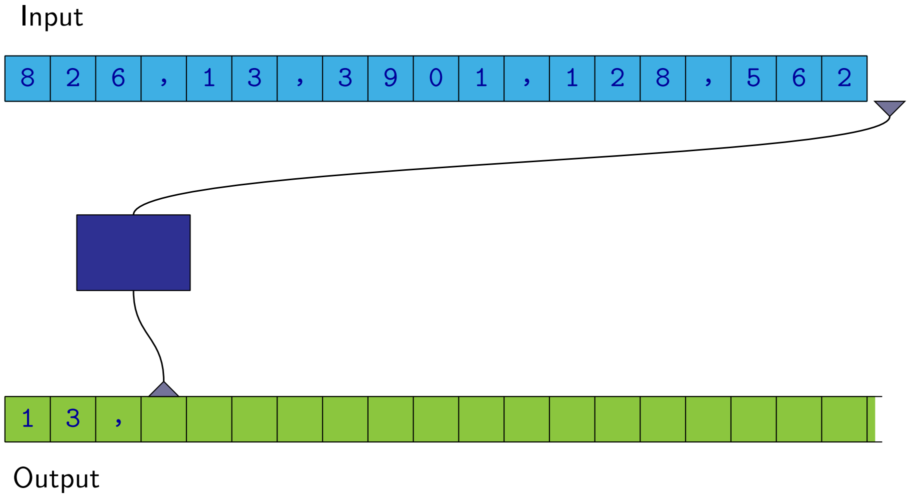

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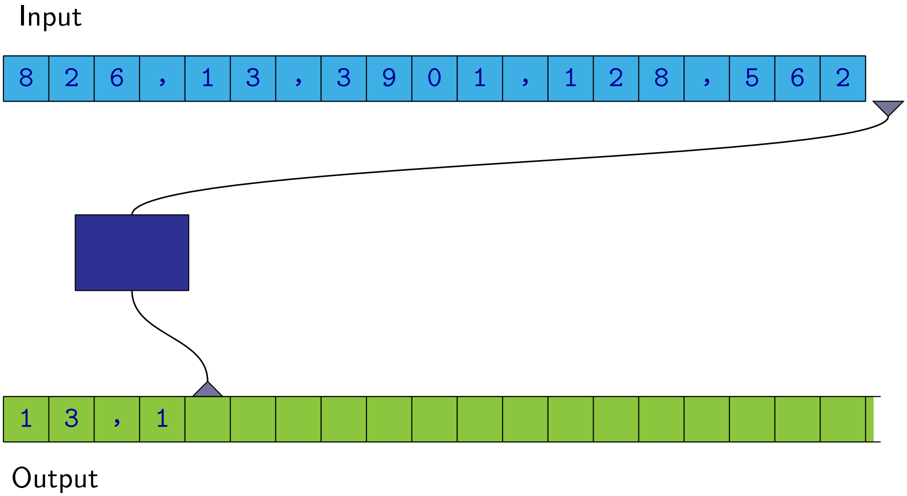

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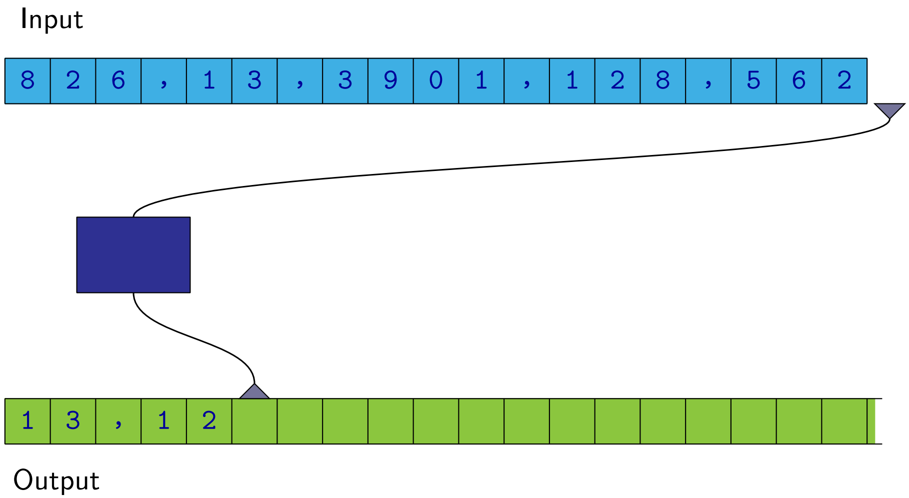

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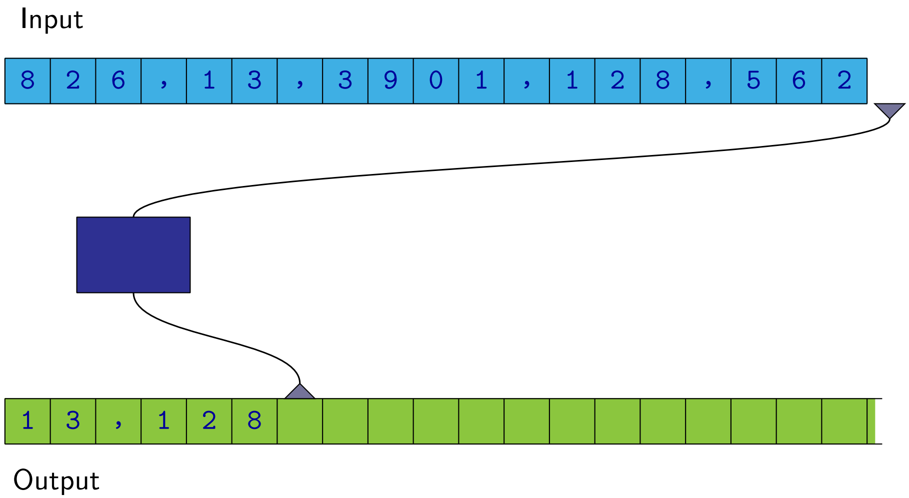

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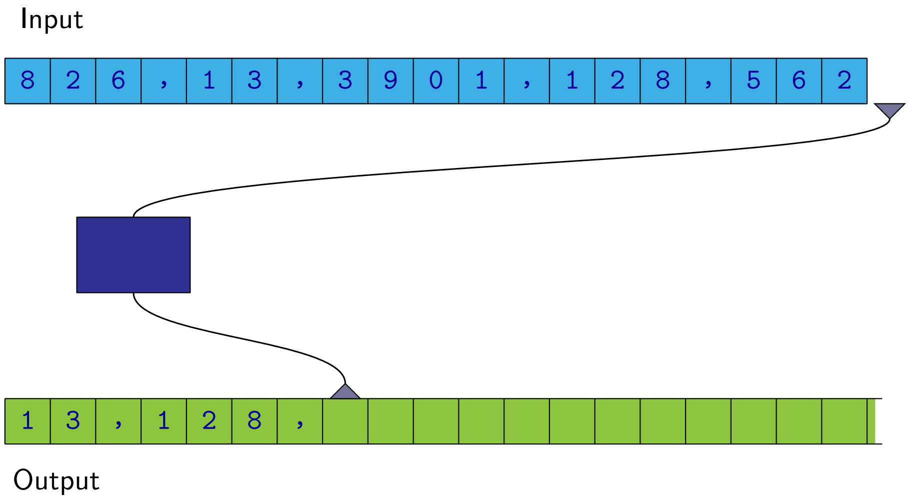

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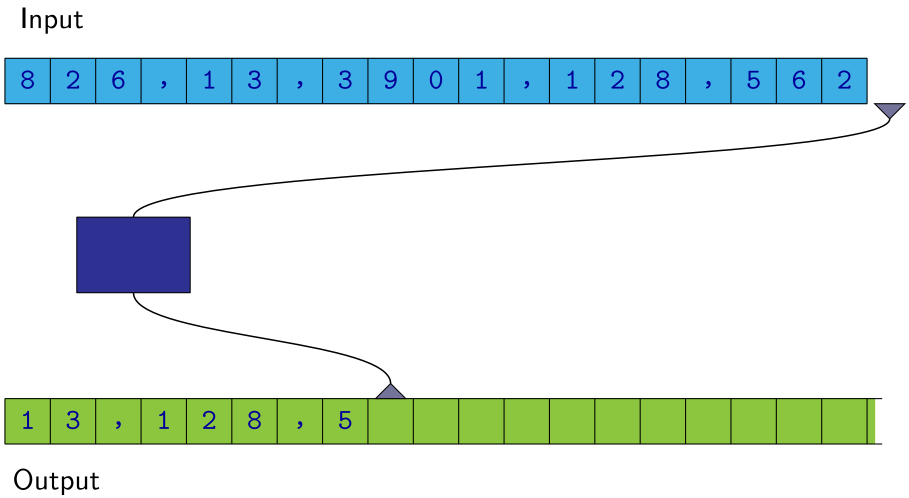

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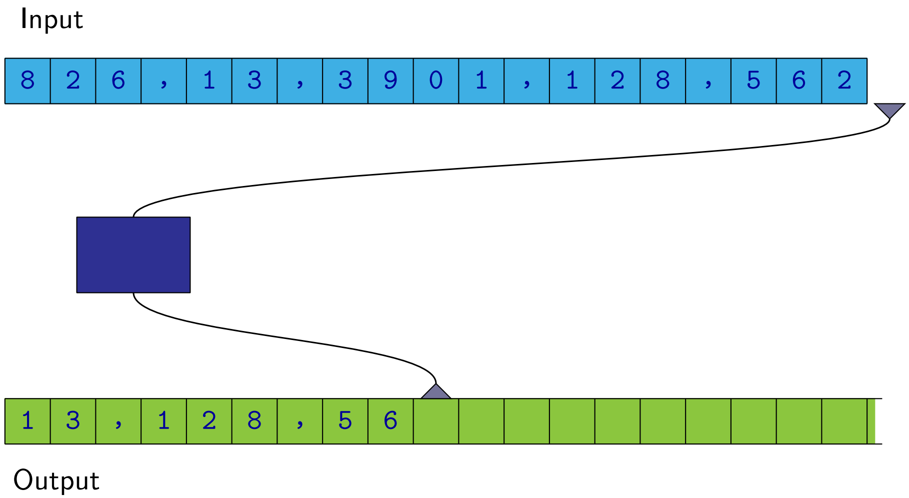

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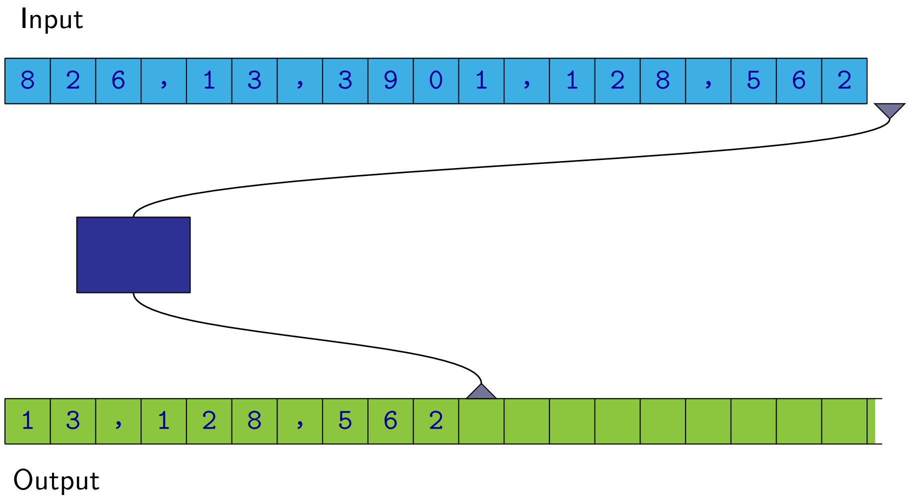

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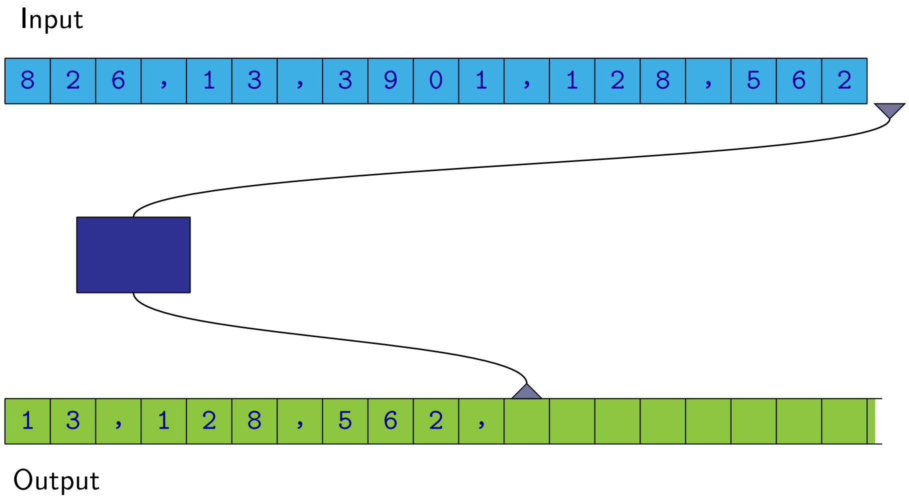

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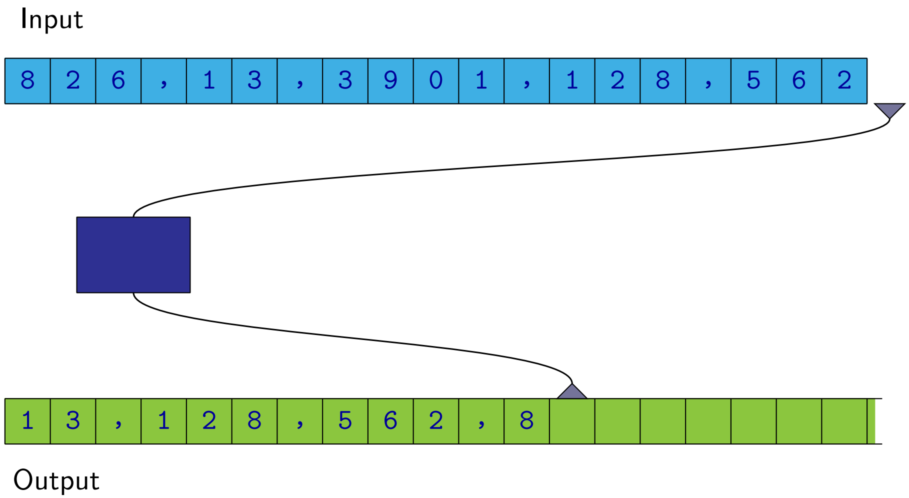

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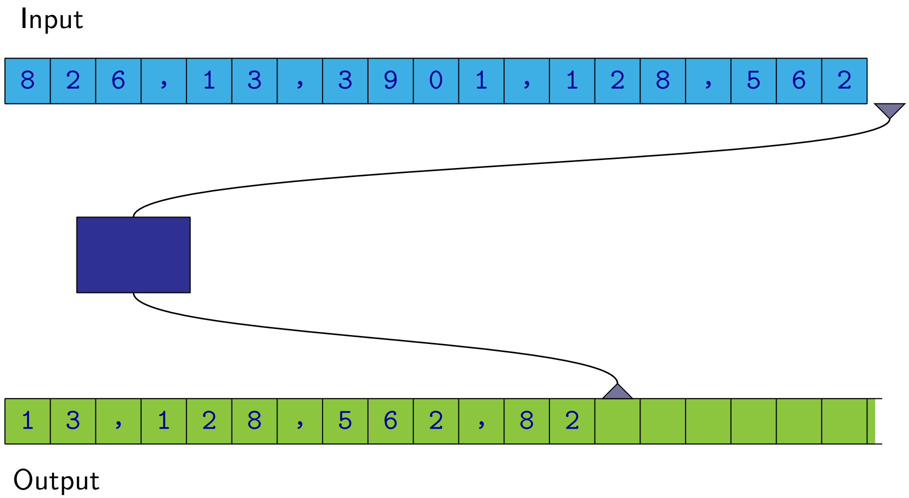

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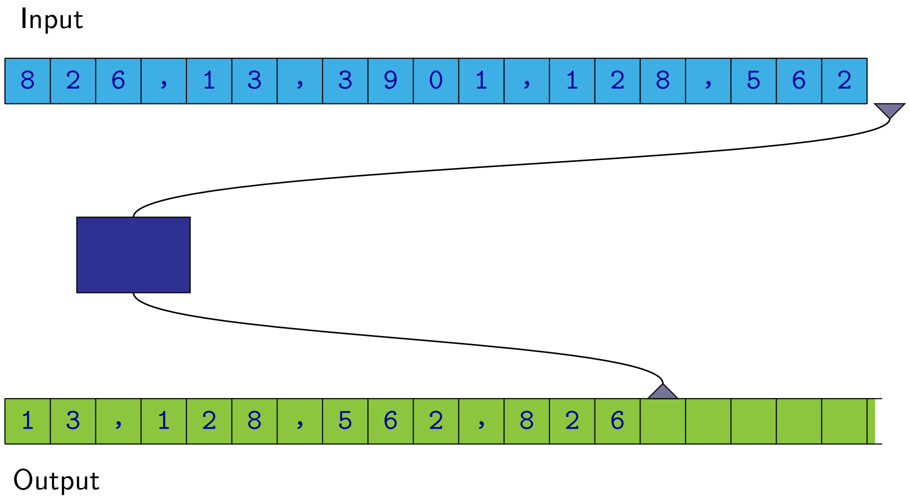

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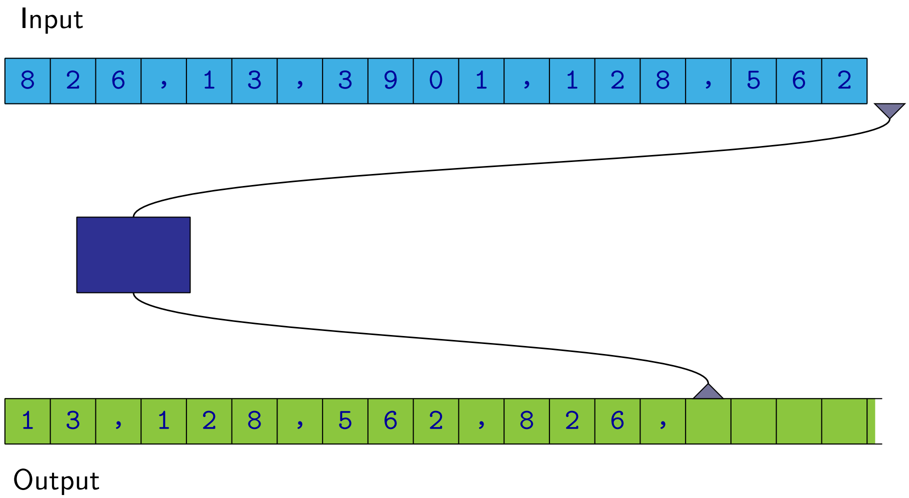
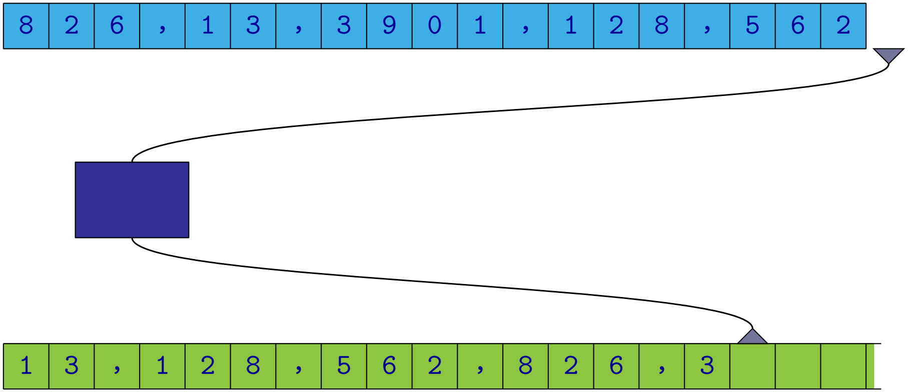


Input/Output Behaviour of an Algorithm

We can assume that the algorithm is executed on a certain type of machine.

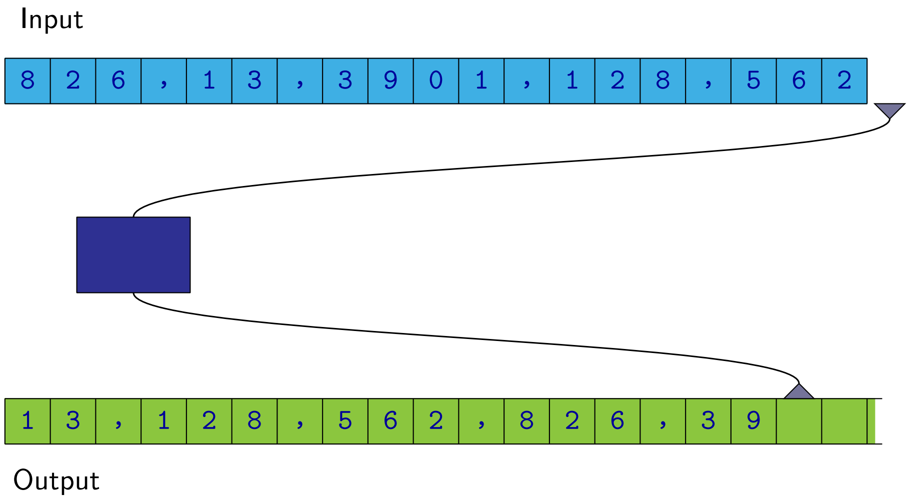
Input



Output

Input/Output Behaviour of an Algorithm

We can assume that the algorithm is executed on a certain type of machine.



Input/Output Behaviour of an Algorithm

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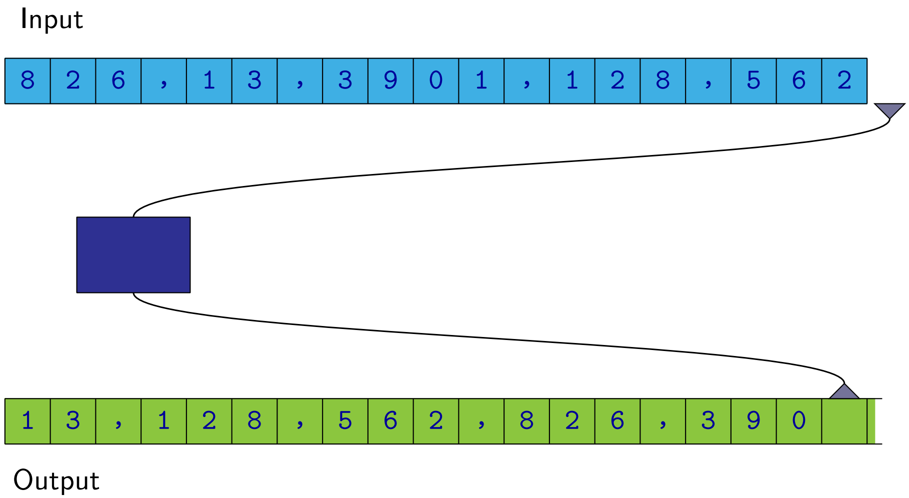

Input/Output Behaviour of an Algorithm

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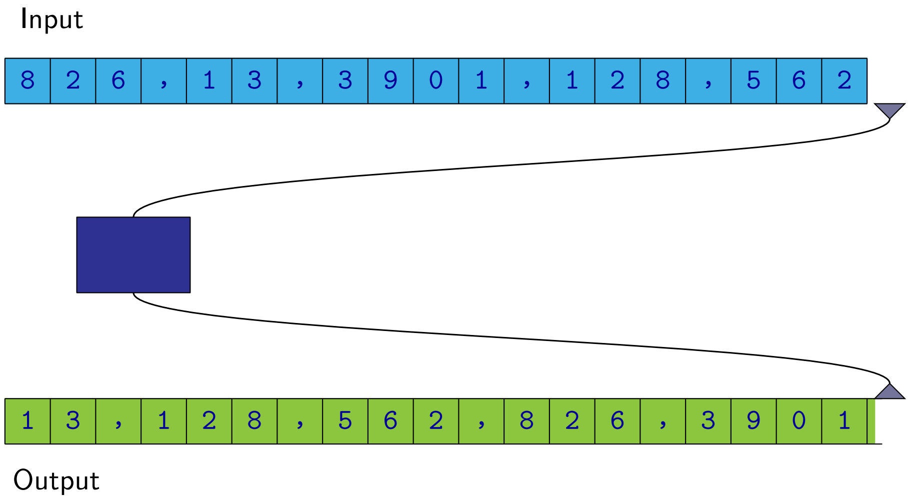
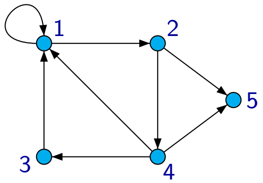


Encoding of Input and Output

Example: If an input for a given problem is graph, it could be represented as a pair of two lists — a list of nodes and a list of edges:

For example, the following graph



could be represented as a word

```
(1,2,3,4,5),((1,2),(2,4),(4,3),(3,1),(1,1),(2,5),(4,5),(4,1))
```

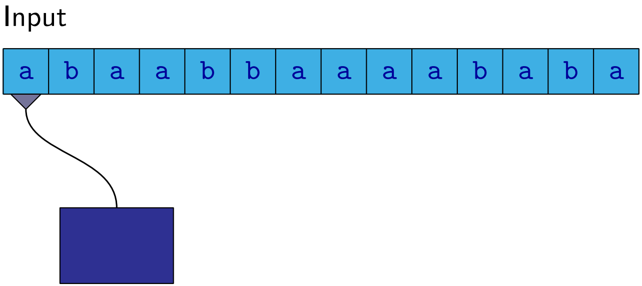
over alphabet $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \texttt{,}, (,)\}$.

Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input:	A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.
Question:	Does the word w contain an even number of occurrences of symbol $\mathtt{b}$?



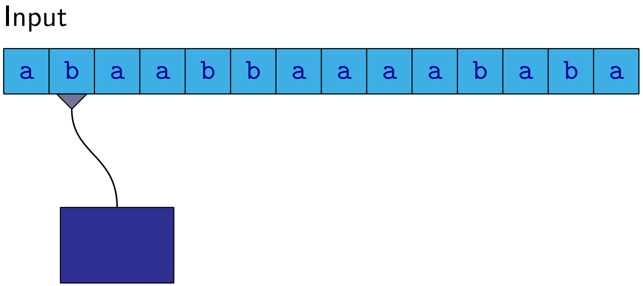
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{a, b\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?



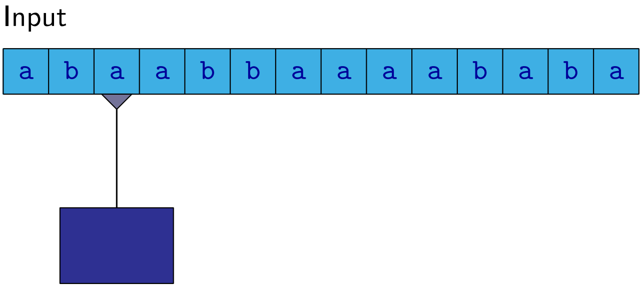
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol `b` ?



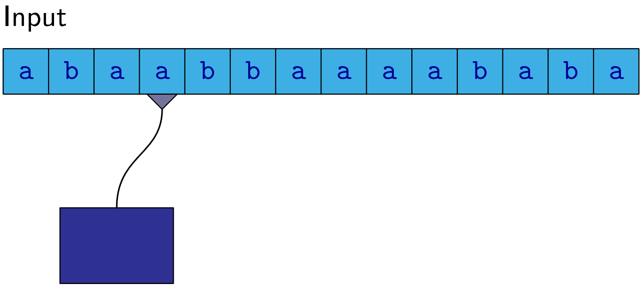
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{a, b\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?



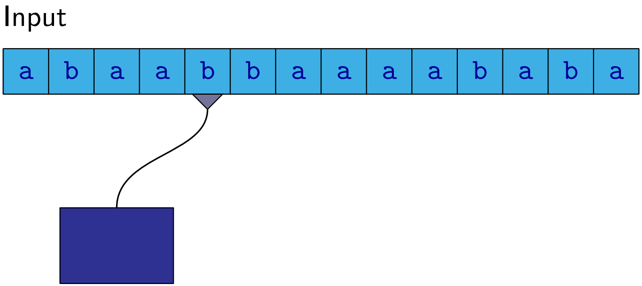
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol $\mathtt{b}$?



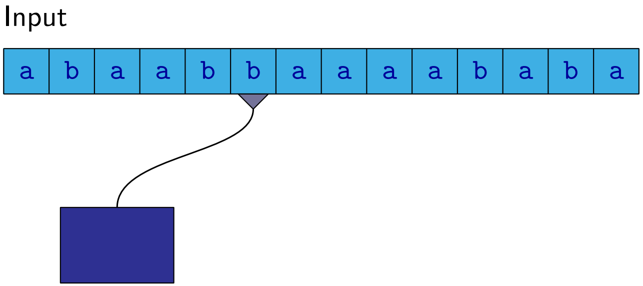
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?



Algorithms for Decision Problems

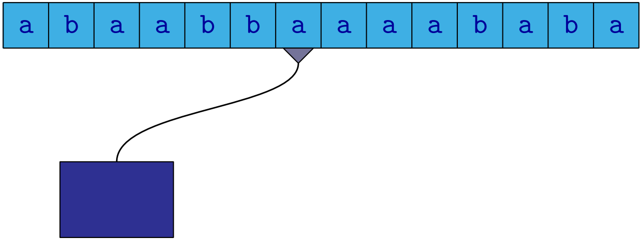
In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\texttt{a}, \texttt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?

Input

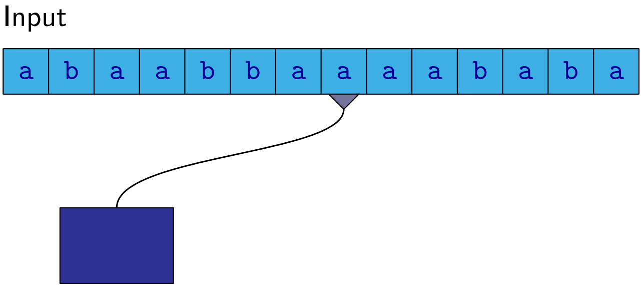


Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input:	A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.
Question:	Does the word w contain an even number of occurrences of symbol $\mathtt{b}$?



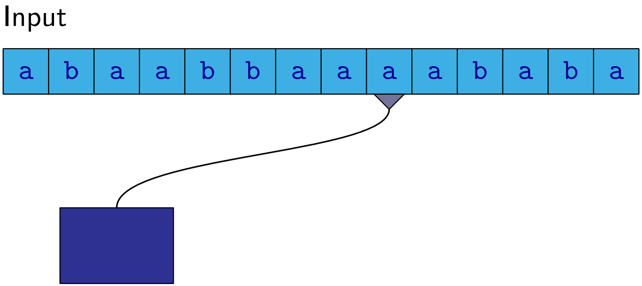
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\texttt{a}, \texttt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol $\texttt{b}$?



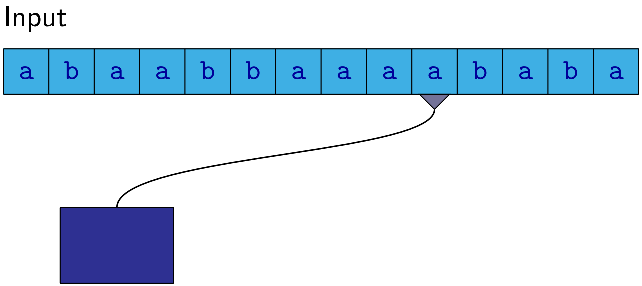
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?



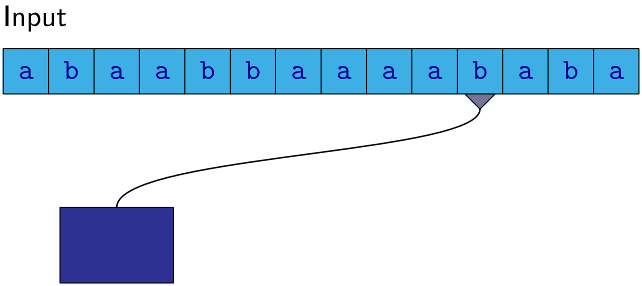
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?



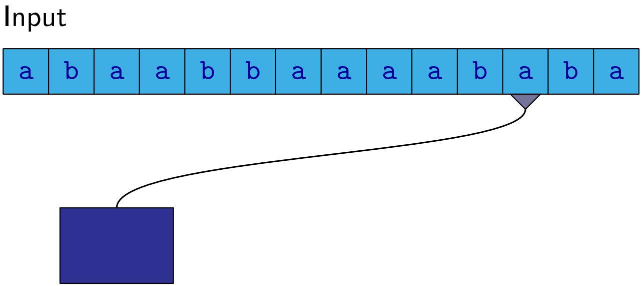
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{a, b\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?



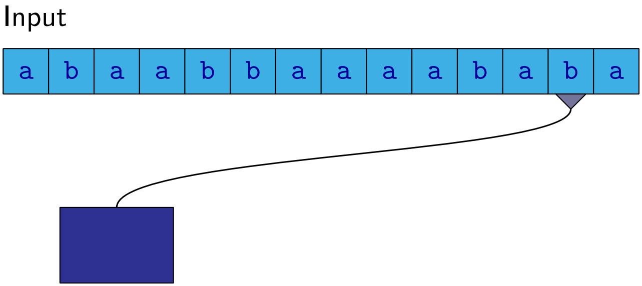
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol $\mathtt{b}$?



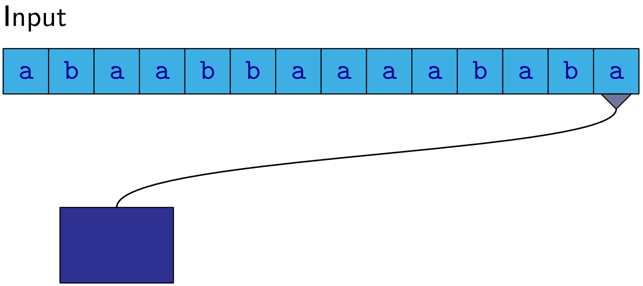
Algorithms for Decision Problems

In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol b ?



Algorithms for Decision Problems

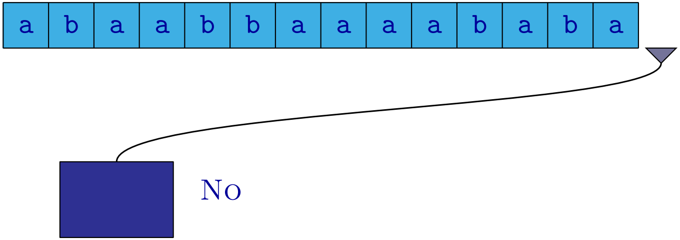
In the case of an algorithm that solves some **decision** problem it is sufficient that the algorithm just provides an answer YES or NO.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol $\mathtt{b}$?

Input



Correspondence between Recognizing Formal Languages and Decision Problems

There is a close correspondence between recognizning words from a given language and decision problems:

- For each language L over some alphabet Σ there is a corresponding decision problem:

Input:	A word w over alphabet Σ.
Question:	Does w belong to L?

- For each decision problem P where inputs are encoded as words over alphabet Σ there is a corresponding language:

The language L containing of exactly those words w over alphabet Σ, for which the answer to the question stated in problem P is "YES".

Correspondence between Recognizing Formal Languages and Decision Problems

Example: The following decision problem can be viewed as the language L given below and vice versa.

Problem

Input: A word w over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

Question: Does the word w contain an even number of occurrences of symbol $\mathtt{b}$?

Language

$L = \{\, w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ contains an even number of occurrences of symbol } \mathtt{b} \,\}$

Models of Computation

We can consider different types of machines that are able to perform an algorithm.

There can be many different kinds of differences between these types of machines:

- what types of instructions they can execute
- what types of dates they can store in their memory and this memory is organised
- ...

Different kinds of such machines are called **models of computation**.

In the case of very simple kinds of such machines they are usually called **automata** in the formal language theory.

In this course we will see several types of such automata.

Models of Computation

For different types of models of computation analyse for example:

- what algorithmic problems can be solved by such machines and what languages they can recognise.
- how efficiently they can execute different algorithms
- how machines of a certain type can simulate the computations of some other type of machines
- how the number of instructions that are executed by the machine in such simulaton grows compared to the original machine
- ...

Formal Languages

Some Basic Concepts

The **set of all words** over alphabet Σ is denoted Σ^*.

The **length of a word** is the number of symbols of the word.
For example, the length of word $\texttt{abaab}$ is 5.

The length of a word w is denoted $|w|$.
For example, if $w = \texttt{abaab}$ then $|w| = 5$.

We denote the number of occurrences of a symbol a in a word w by $|w|_a$.
Example: If $w = \texttt{cabcbba}$ then $|w| = 7$, $|w|_{\texttt{a}} = 2$, $|w|_{\texttt{b}} = 3$, $|w|_{\texttt{c}} = 2$, $|w|_{\texttt{d}} = 0$.

An **empty word** is a word of length 0, i.e., the word containing no symbols.
The empty word is denoted by the letter ε (epsilon) of the Greek alphabet.

$$|\varepsilon| = 0$$

Concatenation of Words

One of operations we can do on words is the operation of **concatenation**:

For example, the concatenation of words `cabc` and `bba` is the word `cabcbba`.

The operation of concatenation is denoted by symbol $\cdot$ (it is similar to multiplication). This symbol can be omitted.

So, for $u, v \in \Sigma^*$, the concatenation of words u and v is written as $u \cdot v$ or just uv.

Example: If $u = \texttt{cabc}$ and $v = \texttt{bba}$, then

$$uv = \texttt{cabcbba}$$

Remark: Formally, the concatenation of words over alphabet Σ is a fuction of type

$$\Sigma^* \times \Sigma^* \to \Sigma^*$$

Concatenation of Words

Concatenation is **associative**, i.e., for every three words u, v, and w we have

$$(u \cdot v) \cdot w = u \cdot (v \cdot w)$$

which means that we can omit parenthesis when we write multiple concatenations. For example, we can write $w_1 \cdot w_2 \cdot w_3 \cdot w_4 \cdot w_5$ instead of $(w_1 \cdot (w_2 \cdot w_3)) \cdot (w_4 \cdot w_5)$.

Word ε is a neutral element for the operation of concatenation, so for every word w we also have:

$$\varepsilon \cdot w = w \cdot \varepsilon = w$$

Remark: It is obvious that if the given alphabet contains at least two different symbols, the operation of concatenation is not commutative, e.g.,

$$\mathtt{a} \cdot \mathtt{b} \neq \mathtt{b} \cdot \mathtt{a}$$

Power of a Word

For arbitrary word $w \in \Sigma^*$ and arbitrary $k \in \mathbb{N}$ we can define word w^k as the word obtained by concatenating k copies of the word w.

Example: For $w = \texttt{abb}$ it is $w^4 = \texttt{abbabbabbabb}$.

Example: Notation $\texttt{a}^5\texttt{b}^3\texttt{a}^4$ denotes word $\texttt{aaaaabbbaaaa}$.

A little bit more formal definition looks as follows:

$$w^0 = \varepsilon, \qquad w^{k+1} = w^k \cdot w \quad \text{for } k \in \mathbb{N}$$

This means

$$\begin{array}{rcl} w^0 & = & \varepsilon \\ w^1 & = & w \\ w^2 & = & w \cdot w \\ w^3 & = & w \cdot w \cdot w \\ w^4 & = & w \cdot w \cdot w \cdot w \\ w^5 & = & w \cdot w \cdot w \cdot w \cdot w \\ & & \cdots \end{array}$$

Reverse of a Word

The **reverse** of a word w is the word w written from backwards (in the opposite order).

The reverse of a word w is denoted w^R.

Example: $w = \mathtt{HELLO}$ $w^R = \mathtt{OLLEH}$

So if $w = a_1 a_2 \cdots a_n$ (where $a_i \in \Sigma$) then $w^R = a_n a_{n-1} \cdots a_1$.

We can define w^R using the following inductively defined function $rev : \Sigma^* \to \Sigma^*$ as the value $rev(w)$.

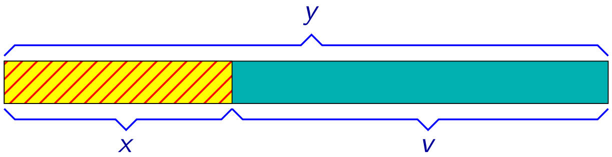
The function rev is defined as follows:

- $rev(\varepsilon) = \varepsilon$
- for $a \in \Sigma$ and $w \in \Sigma^*$ it holds that $rev(a \cdot w) = rev(w) \cdot a$

Prefix of a Word

Definition

A word x is a **prefix** of a word y, if there exists a word v such that $y = xv$.

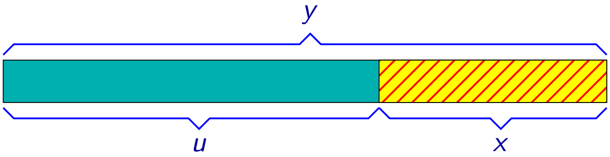


Example: Prefixes of the word `abaab` are ε, `a`, `ab`, `aba`, `abaa`, `abaab`.

Suffix of a Word

Definition

A word x is a **suffix** of a word y, if there exists a word u such that $y = ux$.

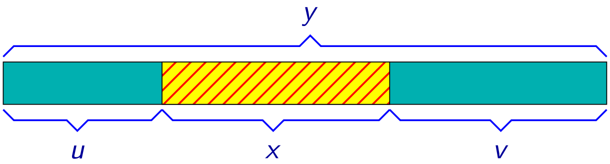


Example: Suffixes of the word `abaab` are ε, `b`, `ab`, `aab`, `baab`, `abaab`.

Subword

Definition

A word x is a **subword** of a word y, if there exist words u and v such that $y = uxv$.



Example: Subwords of the word `abaab` are ε, `a`, `b`, `ab`, `ba`, `aa`, `aba`, `baa`, `aab`, `abaa`, `baab`, `abaab`.

Order on Words

Let us assume some (linear) order $<$ on the symbols of alphabet Σ, i.e., if $\Sigma = \{a_1, a_2, \ldots, a_n\}$ then

$$a_1 < a_2 < \ldots < a_n \,.$$

Example: $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ with $\mathtt{a} < \mathtt{b} < \mathtt{c}$.

The following (linear) order $<_L$ can be defined on Σ^*:
$x <_L y$ iff:

- $|x| < |y|$, or
- $|x| = |y|$ there exist words $u, v, w \in \Sigma^*$ and symbols $a, b \in \Sigma$ such that

$$x = uav \qquad y = ubw \qquad a < b$$

Informally, we can say that in order $<_L$ we order words according to their length, and in case of the same length we order them lexicographically.

Order on Words

All words over alphabet Σ can be ordered by $<_L$ into a sequence

$$w_0, w_1, w_2, \ldots$$

where every word $w \in \Sigma^*$ occurs exactly once, and where for each $i, j \in \mathbb{N}$ it holds that $w_i <_L w_j$ iff $i < j$.

Example: For alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ (where $\mathtt{a} < \mathtt{b} < \mathtt{c}$) , the initial part of the sequence looks as follows:

$$\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{aa}, \mathtt{ab}, \mathtt{ac}, \mathtt{ba}, \mathtt{bb}, \mathtt{bc}, \mathtt{ca}, \mathtt{cb}, \mathtt{cc}, \mathtt{aaa}, \mathtt{aab}, \mathtt{aac}, \mathtt{aba}, \mathtt{abb}, \mathtt{abc}, \ldots$$

For example, when we talk about the first ten words of a language $L \subseteq \Sigma^*$, we mean ten words that belong to language L and that are smallest of all words of L according to order $<_L$.

Operations on Languages

Let us say we have already described some languages. We can create new languages from these languages using different **operations on languages**.

So a description of a complicated language can be decomposed in such a way that it is described a result of an application of some operations on some simpler languages.

Examples of important operations on languages:

- union
- intersection
- complement
- concatenation
- iteration
- ...

Remark: It is assumed the languages involved in these operations use the same alphabet Σ.

Set Operations on Languages

Since languages are sets, we can apply any set operations to them:

Union – $L_1 \cup L_2$ is the language consisting of the words belonging to language L_1 or to language L_2 (or to both of them).

Intersection – $L_1 \cap L_2$ is the language consisting of the words belonging to language L_1 and also to language L_2.

Complement – $\overline{L_1}$ is the language containing those words from Σ^* that do not belong to L_1.

Difference – $L_1 - L_2$ is the language containing those words of L_1 that do not belong to L_2.

Set Operations on Languages

Formally:

Union: $L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \vee w \in L_2\}$

Intersection: $L_1 \cap L_2 = \{w \in \Sigma^* \mid w \in L_1 \wedge w \in L_2\}$

Complement: $\overline{L_1} = \{w \in \Sigma^* \mid w \notin L_1\}$

Difference: $L_1 - L_2 = \{w \in \Sigma^* \mid w \in L_1 \wedge w \notin L_2\}$

Remark: We assume that $L_1, L_2 \subseteq \Sigma^*$ for some given alphabet Σ.

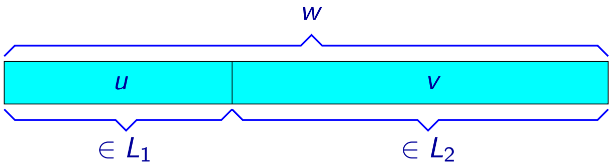
Concatenation of Languages

Definition

Concatenation of languages L_1 and L_2, where $L_1, L_2 \subseteq \Sigma^*$, is the language $L \subseteq \Sigma^*$ such that for each $w \in \Sigma^*$ it holds that

$$w \in L \iff (\exists u \in L_1)(\exists v \in L_2)(w = u \cdot v)$$

The concatenation of languages L_1 and L_2 is denoted $L_1 \cdot L_2$.



Concatenation of Languages

Example:

$$\begin{aligned} L_1 &= \{\mathtt{abb}, \mathtt{ba}\} \\ L_2 &= \{\mathtt{a}, \mathtt{ab}, \mathtt{bbb}\} \end{aligned}$$

The language $L_1 \cdot L_2$ contains the following words:

`abba` `abbab` `abbbbb` `baa` `baab` `babbb`

Remark: Note that the concatenation of languages is associative, i.e., for arbitrary languages L_1, L_2, L_3 it holds that:

$$L_1 \cdot (L_2 \cdot L_3) = (L_1 \cdot L_2) \cdot L_3$$

Power of a Language

Notation L^k, where $L \subseteq \Sigma^*$ and $k \in \mathbb{N}$, denotes the concatenation of the form

$$L \cdot L \cdot \quad \cdots \quad \cdot L$$

where the language L occurs k times, i.e.,

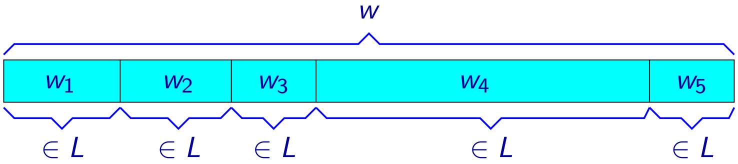
$$\begin{array}{lcl} L^0 & = & \{\varepsilon\} \\ L^1 & = & L \\ L^2 & = & L \cdot L \\ L^3 & = & L \cdot L \cdot L \\ L^4 & = & L \cdot L \cdot L \cdot L \\ L^5 & = & L \cdot L \cdot L \cdot L \cdot L \\ & & \ldots \end{array}$$

Example: For $L = \{\mathtt{aa}, \mathtt{b}\}$, the language L^3 contains the following words:

`aaaaaa` `aaaab` `aabaa` `aabb` `baaaa` `baab` `bbaa` `bbb`

Power of a Language

Example: A word in language L^5 is created by concatenating five words from language L:



Formally, the k-th power of a language L, denoted L^k can be defined using the following inductive definition:

$$L^0 = \{\varepsilon\}, \qquad L^{k+1} = L^k \cdot L \quad \text{for } k \in \mathbb{N}$$

Iteration of a Language

The **iteration of a language** L, denoted L^*, is the language consisting of words created by concatenation of some arbitrary number of words from language L.

I.e., a word w belongs to L^* iff there exists a sequence $w_1, w_2, \ldots, w_n$ of words from language L such that

$$w = w_1 w_2 \cdots w_n \,.$$

Example: $L = \{\mathrm{aa}, \mathrm{b}\}$

$$L^* = \{\varepsilon, \mathrm{aa}, \mathrm{b}, \mathrm{aaaa}, \mathrm{aab}, \mathrm{baa}, \mathrm{bb}, \mathrm{aaaaaa}, \mathrm{aaaab}, \mathrm{aabaa}, \mathrm{aabb}, \ldots\}$$

Remark: The number of concatenated words can be 0, which means that $\varepsilon \in L^*$ always holds (it does not matter if $\varepsilon \in L$ or not).

Iteration of a Language

Formally, the language L^* can be defined as the union of all powers of language L. I.e., a word w belongs to the language L^* iff if there exists $k \in \mathbb{N}$ such that $w \in L^k$:

Definition

The **iteration of a language** L is the language

$$L^* = \bigcup_{k \geq 0} L^k$$

Remark:

$$\bigcup_{k \geq 0} L^k = L^0 \cup L^1 \cup L^2 \cup L^3 \cup \cdots$$

Iteration of a Language

Notation L^+ denotes the language consinsting of those words that can be created as a concatenation of a non-zero number of words from language L.

So it holds that

$$L^+ = L^1 \cup L^2 \cup L^3 \cup \cdots$$

Formally, the language L^+ can be defined as follows:

$$L^+ = L \cdot L^*$$

Reverse

The **reverse** of a language L is the language consisting of reverses of all words of L.

Reverse of a language L is denoted L^R.

$$L^R = \{w^R \mid w \in L\}$$

Example: $L = \{\mathtt{ab}, \mathtt{baaba}, \mathtt{aaab}\}$

$L^R = \{\mathtt{ba}, \mathtt{abaab}, \mathtt{baaa}\}$

Left and Right Quotient

- For a word $w \in \Sigma^*$ and language $L \subseteq \Sigma^*$, the **left quotient of the language L with the word w**, denoted $w \backslash L$, is defined as follows:

$$w \backslash L = \{ v \in \Sigma^* \mid wv \in L \}$$

- For a pair of languages $L_1, L_2 \subseteq \Sigma^*$, the **left quotient** is defined as follows:

$$L_1 \backslash L_2 = \{ v \in \Sigma^* \mid \exists w \in L_1 : wv \in L_2 \}$$

- Analogous definitions of the **right quotient** look as follows:

$$L / w = \{ v \in \Sigma^* \mid vw \in L \}$$

$$L_1 / L_2 = \{ v \in \Sigma^* \mid \exists w \in L_2 : vw \in L_1 \}$$

Some Properties of Operations on Languages

$$\begin{aligned}
L_1 \cup (L_2 \cup L_3) &= (L_1 \cup L_2) \cup L_3 \\
L_1 \cup L_2 &= L_2 \cup L_1 \\
L_1 \cup L_1 &= L_1 \\
L_1 \cup \emptyset &= L_1
\end{aligned}$$

$$\begin{aligned}
L_1 \cap (L_2 \cap L_3) &= (L_1 \cap L_2) \cap L_3 \\
L_1 \cap L_2 &= L_2 \cap L_1 \\
L_1 \cap L_1 &= L_1 \\
L_1 \cap \emptyset &= \emptyset
\end{aligned}$$

$$\begin{aligned}
L_1 \cdot (L_2 \cdot L_3) &= (L_1 \cdot L_2) \cdot L_3 \\
L_1 \cdot \{\varepsilon\} &= L_1 \\
\{\varepsilon\} \cdot L_1 &= L_1 \\
L_1 \cdot \emptyset &= \emptyset \\
\emptyset \cdot L_1 &= \emptyset
\end{aligned}$$

Some Properties of Operations on Languages

$$\begin{array}{rcl}
L_1 \cdot (L_2 \cup L_3) & = & (L_1 \cdot L_2) \cup (L_1 \cdot L_3) \\
(L_1 \cup L_2) \cdot L_3 & = & (L_1 \cdot L_3) \cup (L_2 \cdot L_3) \\
(L_1^*)^* & = & L_1^* \\
\emptyset^* & = & \{\varepsilon\} \\
L_1^* & = & \{\varepsilon\} \cup (L_1 \cdot L_1^*) \\
L_1^* & = & \{\varepsilon\} \cup (L_1^* \cdot L_1) \\
(L_1 \cup L_2)^* & = & L_1^* \cdot (L_2 \cdot L_1^*)^* \\
(L_1 \cdot L_2)^R & = & L_2^R \cdot L_1^R
\end{array}$$