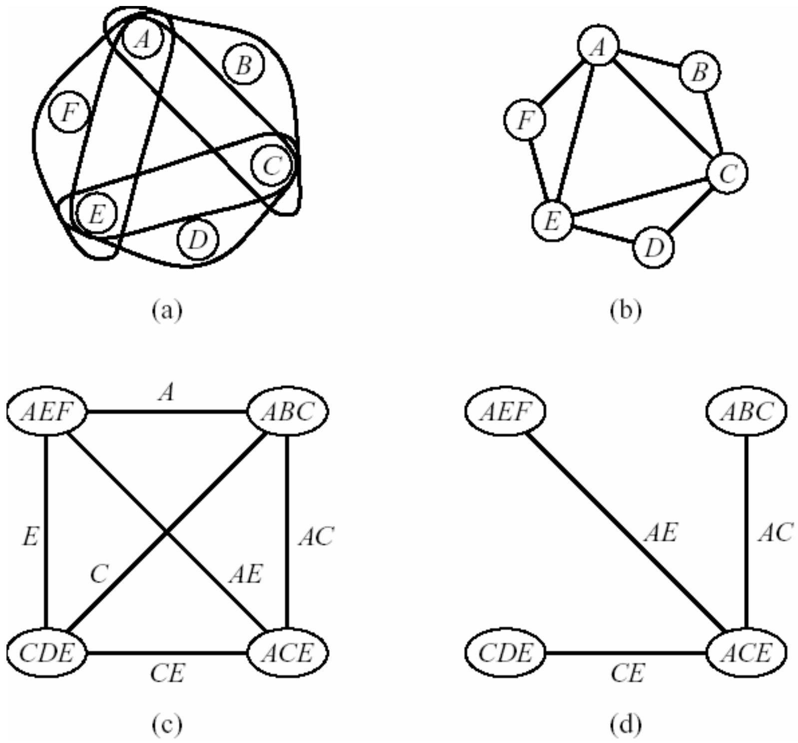
Tree Decomposition methods

Chapter 9

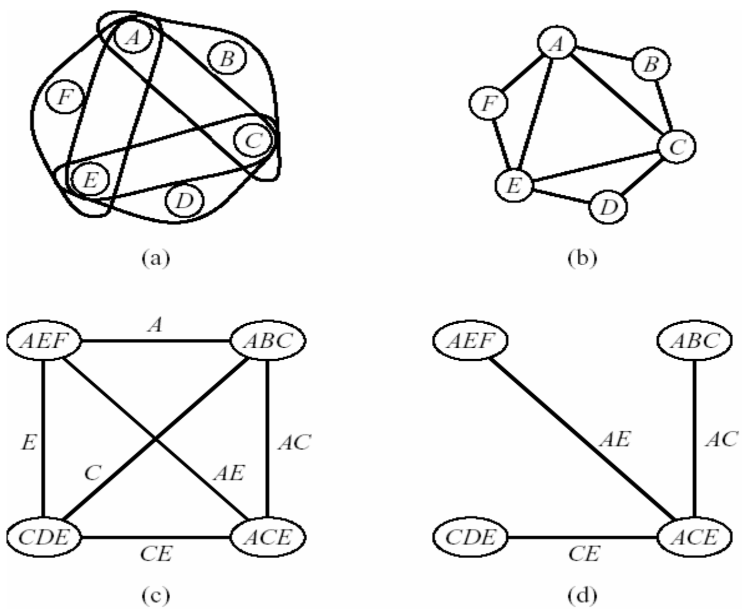
Graph concepts reviews: Hyper graphs and dual graphs

- **A hypergraph** is $H = (V,S)$, $V= \{v_1,..,v_n\}$ and a set of subsets **Hyperegdes**: $S=\{S_1, ..., S_l \}$.
- **Dual graphs** of a hypergaph: The nodes are the hyperedges and a pair of nodes is connected if they share vertices in V. The arc is labeled by the shared vertices.
- **A primal graph** of a hypergraph $H = (V,S)$ has V as its nodes, and any two nodes are connected by an arc if they appear in the same hyperedge.
- if all the constraints of a network R are binary, then its hypergraph is identical to its primal graph.



Acyclic Networks

- **The running intersection property (connectedness):** An arc can be removed from the dual graph if the variables labeling the arcs are shared along an alternative path between the two endpoints.
- **Join graph**: An arc subgraph of the dual graph that satisfies the connectedness property.
- **Join-tree:** a join-graph with no cycles
- **Hypertree:** A hypergraph whose dual graph has a join-tree.
- **Acyclic network:** is one whose hypergraph is a hypertree.



Solving acyclic networks

- Algorithm ***acyclic-solving*** applies a tree algorithm to the join-tree. It applies directional relational arc-consistency from leaves to root.

- **Complexity**: acyclic-solving is $O(r\ l\ \log l)$ steps, where r is the number of constraints and l bounds the number of tuples in each constraint relation

Example

- Constraints are:
- *R_{ABC} = R_{AEF} = R_{CDE} = {(0,0,1) (0,1,0)(1,0,0)}*
- *R_{ACE} = { (1,1,0) (0,1,1) (1,0,1) }.*

- d= (R_{ACE},R_{CDE},R_{AEF},R_{ABC}).
  - When processing R_{ABC}, its parent relation is R_{ACE};

$$R_{ACE} = \pi_{ACE}(R_{ACE} \otimes R_{ABC}) = \{(0,1,1)(1,0,1)\}$$

  - processing R_{AEF} we generate relation

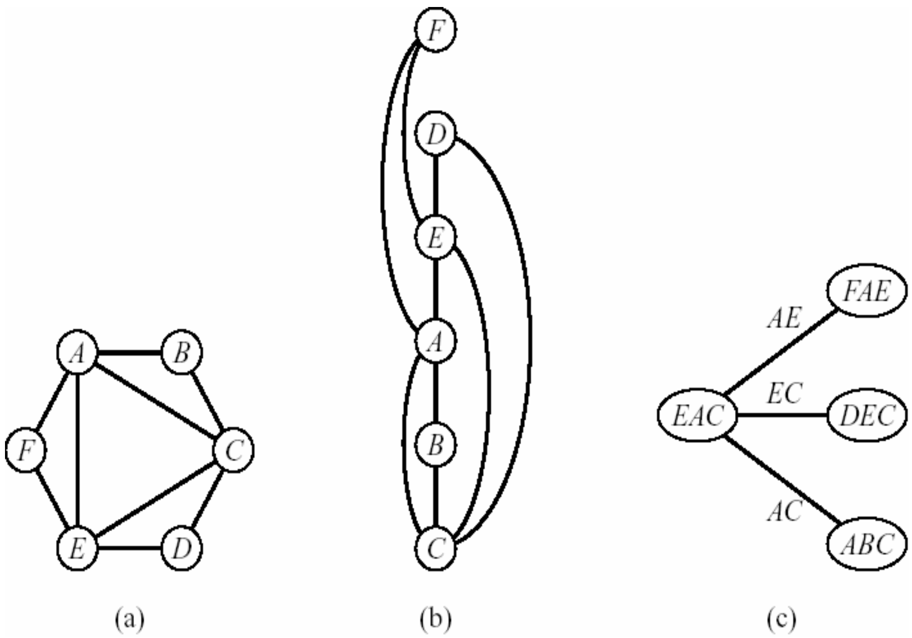
$$R_{ACE} = \pi_{ACE}(R_{ACE} \otimes R_{AEF}) = \{(0,1,1)\}$$

    - processing R_{CDE} we generate:
    - R_{ACE} = \pi_{ACE} (R_{ACE} x R_{CDE}) = {(0,1,1)}.
- A solution is generated by picking the only allowed tuple for R_{ACE}, A=0,C=1,E=1, extending it with a value for D that satisfies R_{CDE}, which is only D=0, and then similarly extending the assignment to F=0 and B=0, to satisfy R_{AEF} and R_{ABC}.

Recognizing acyclic networks

- **Dual-based recognition**:
  - Perform maximal spanning tree over the dual graph and check connectedness of the resulting tree.
  - Dual-acyclicity complexity is O(e^3)
- **Primal-based recognition**:
  - **Theorem** (Maier 83): A hypergraph has a join-tree iff its primal graph is chordal and conformal.
  - A chordal primal graph is **conformal** relative to a constraint hypergraph iff there is a one-to-one mapping between maximal cliques and scopes of constraints.

Primal-based recognition



- Check chordality using max-cardinality ordering.
- Test conformality
- Create a join-tree: connect every clique to an earlier clique sharing maximal number of variables.

Tree-based clustering

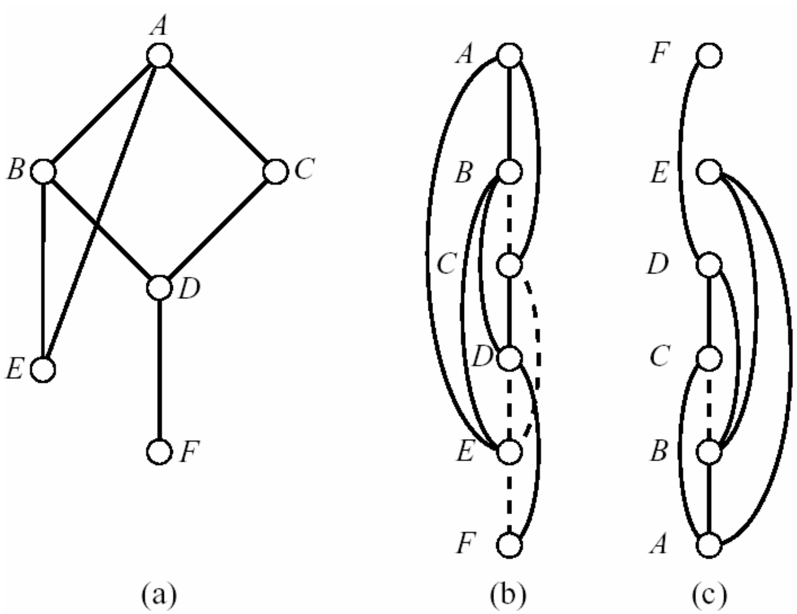
- Convert a constraint problem to an acyclic-one: group subset of constraints to clusters until we get an acyclic problem.
- Hypertree embedding of a hypergraph H = (X,H) is a hypertree S = (X, S) s.t., for every h in H there is h_1 in S s.t. h is included in h_1.
- This yield algorithm join-tree clustering

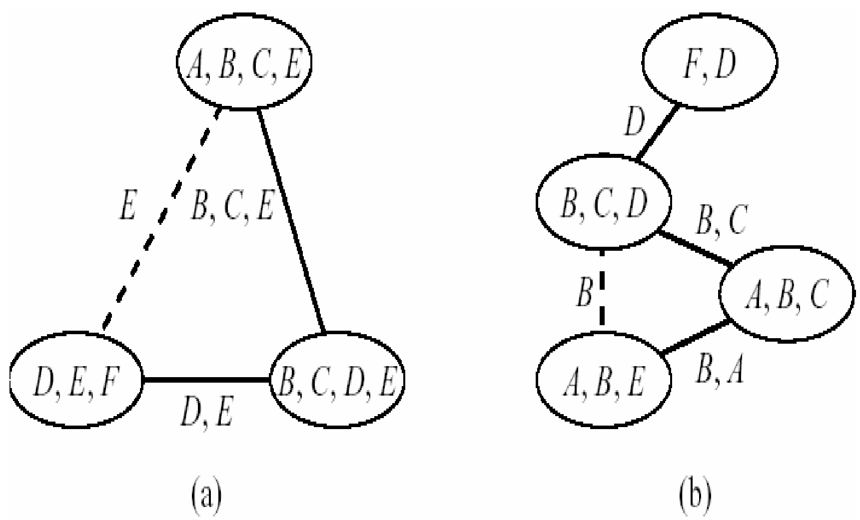
Join-tree clustering

- **Input**: A constraint problem R =(X,D,C) and its primal graph G = (X,E).
- **Output**: An equivalent acyclic constraint problem and its join-tree*: T*= (X,D, {C '})
- 1. Select an d = (x_1,...,x_n)
- 2. **Triangulation**(create the induced graph along d and call it G^*:)
- for j=n to 1 by -1 do
- E ← E U {(i,k)| (i,j) in E,(k,j) in E }
- 3. **Create a join-tree of the induced graph G^*:**
- a. Identify all maximal cliques (each variable and its parents is a clique).
- Let C_1,...,C_t be all such cliques,
- b. Create a tree-structure T over the cliques:
- Connect each C_{i} to a C_{j} (j < I) with whom it shares largest subset of variables.
- 4. Place each input constraint in one clique containing its scope, and let
- P_i be the constraint subproblem associated with C_i.
- 5. Solve P_i and let {R'}_i $ be its set of solutions.
- 6. Return C' = {R'}_1,..., {R'}_t
- the new set of constraints and their join-tree, T.

- Size of maximal clique - 1 is the Induced width.

Example of tree-clustering

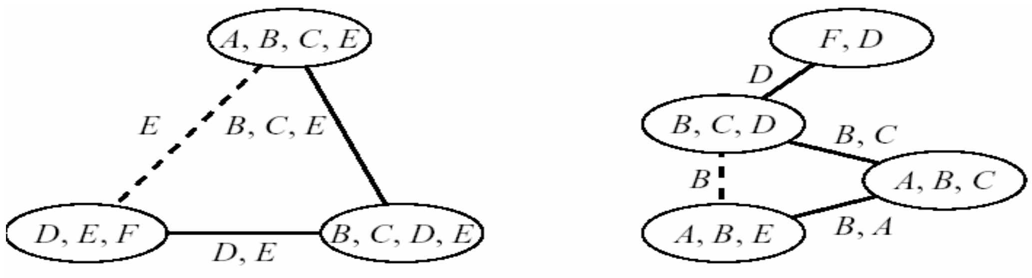




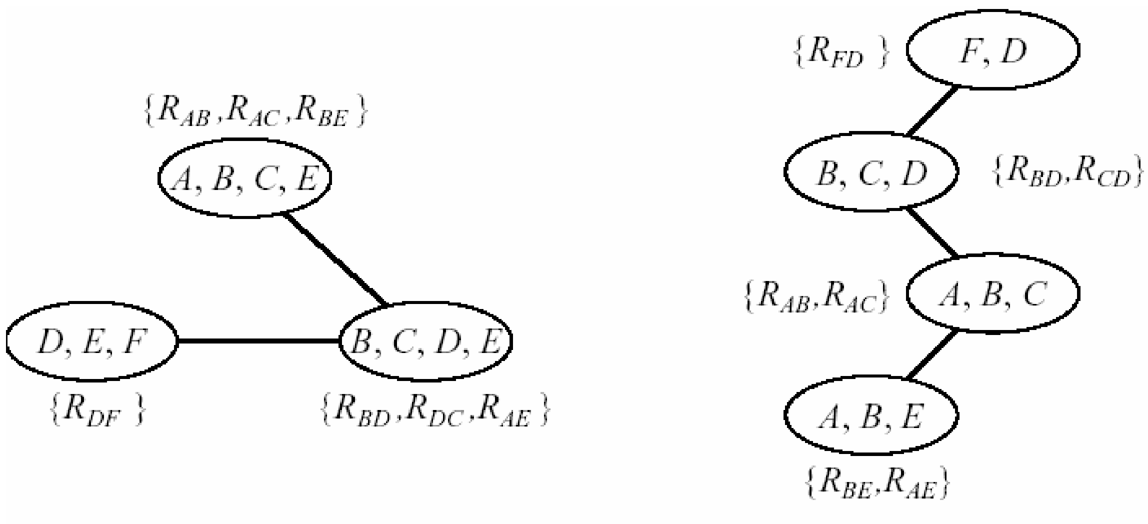
Unifying tree-decompositions

- A tree-decomposition of R = (X, D, C) is a triple $<T, \chi, \psi>$ where $T = <V, E>$ is a tree, and χ and ψ are sets of functions .
  - For each constraint $R_i \in C$ there is at least one vertex v in T such that $R_i \in \psi(v)$ and $scope(R_i) \subseteq \chi(v)$
  - For each variable x in X, the set $\{v \in V \mid x \in \chi(v)\}$ induces a connected subtree of T. (This is the connectedness property.)
- **tree-width** = max number of vars in a cluster
- **hyper-width** = is max functions in a cluster
- the **separator of u and v**: the intersection between variables in u and v.

Example of two join-trees again



(a) (b)

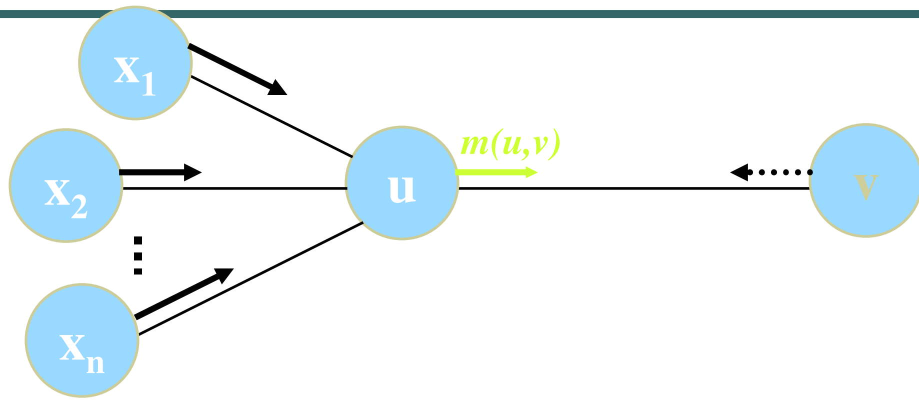


(a) (b)

Cluster Tree Elimination

- Cluster Tree Elimination *(CTE)* works by passing messages along a tree-decomposition
- Basic idea:
  - Each node sends one message to each of its neighbors
  - Node *u* sends a message to its neighbor *v* only when *u* received messages from all its other neighbors

Constraint Propagation

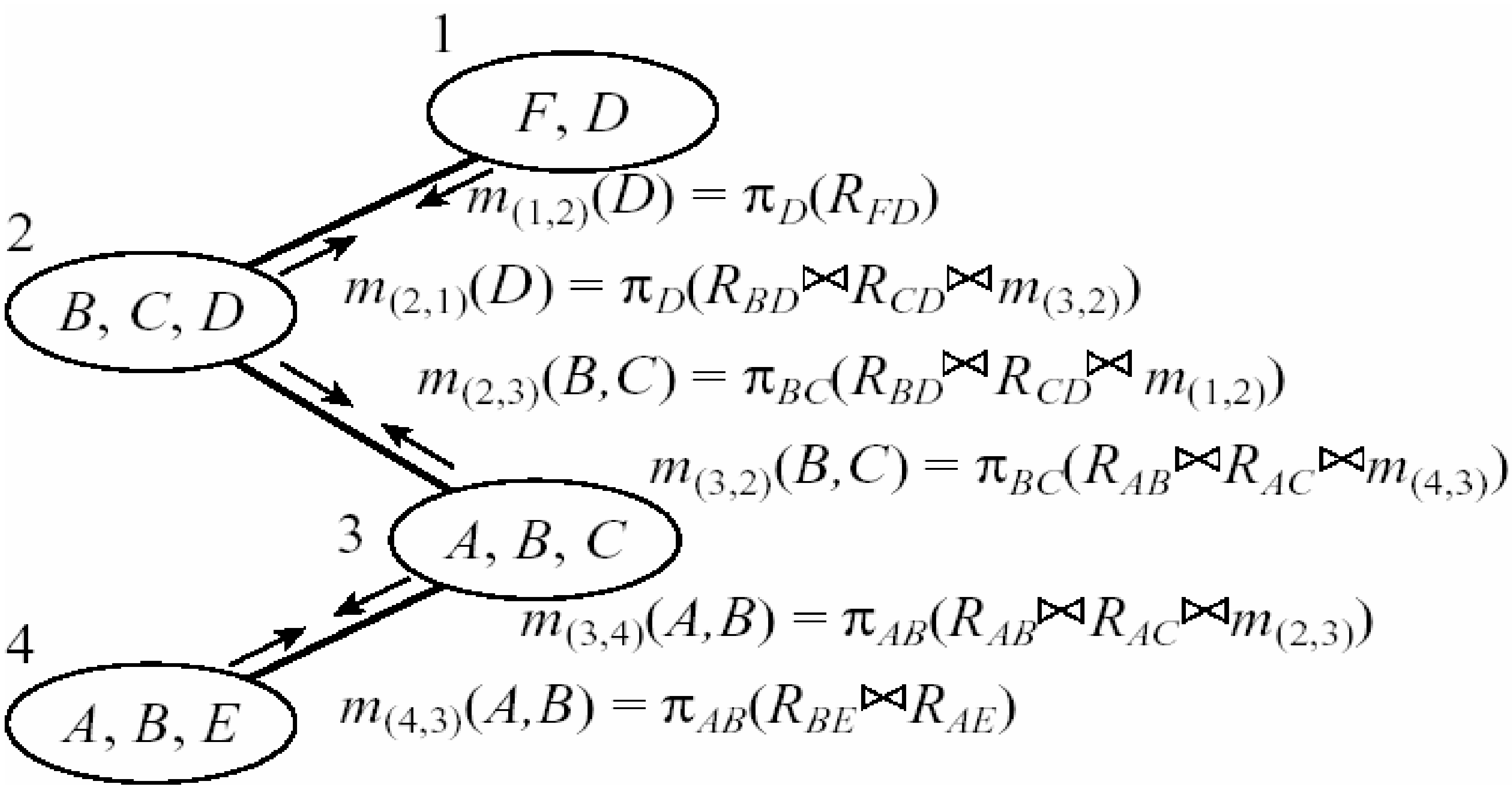


$$cluster(u) = \psi(u) \cup \{m(x_1,u), m(x_2,u), \ldots, m(x_n,u), m(v,u)\}$$

Compute the message :

$$m_{(u,v)} = \pi_{sep(u,v)}(\otimes_{R_i \in cluster(u)} R_i)$$

Example of CTE message propagation



Distributed relational arc-consistency example

The message that R2 sends to R1 is

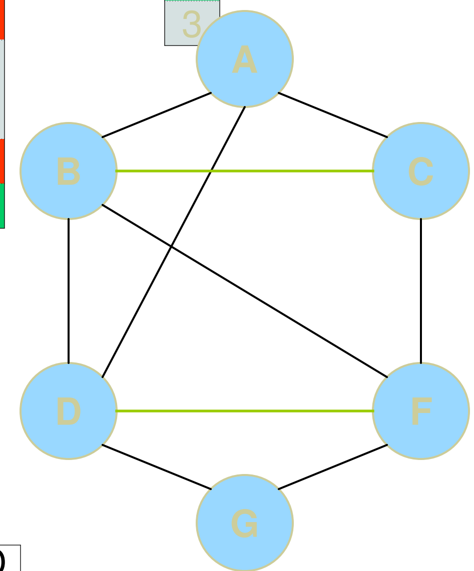
R1 updates its relation and domains and sends messages to neighbors

R_1

A
1
2
3

R_2

A	B
1	2
1	3
2	1
2	3
3	1
3	2



R_3

A	C
1	2
3	2

R_5

B	C	F
1	2	3
3	2	1

R_4

A	B	D
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

R_6

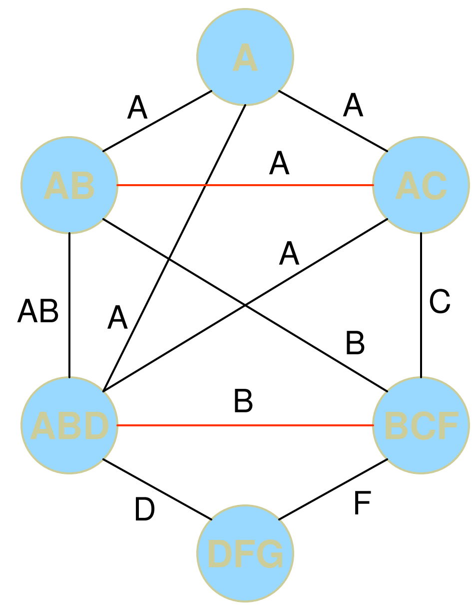
D	F	G
1	2	3
2	1	3

Distributed Arc-Consistency

DR-AC can be applied to the dual problem of any constraint network.

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

$$D_i \leftarrow D_i \cap (\bowtie_{k \in ne(i)} D_k^i) \qquad (2)$$



b) Constraint network

DR-AC on a dual join-graph

R_1

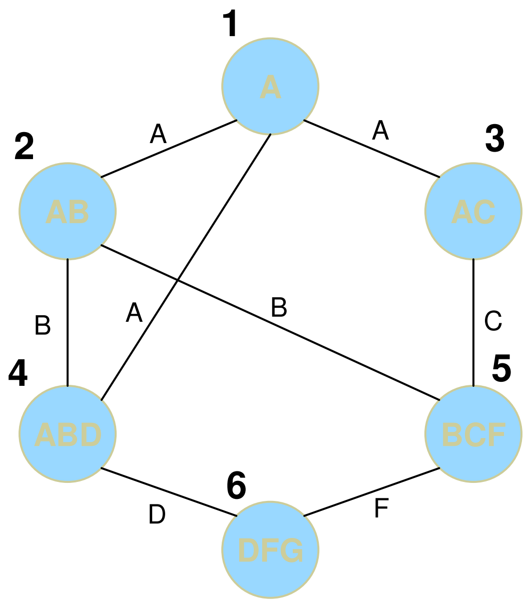
A
1
2
3

R_2

A	B
1	2
1	3
2	1
2	3
3	1
3	2

R_3

A	C
1	2
3	2



R_4

A	B	D
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

R_5

B	C	F
1	2	3
3	2	1

R_6

D	F	G
1	2	3
2	1	3

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

Iteration 1

R_1

A
1
2
3

h_2^1

A
1
2
3

h_3^1

A
1
3

h_4^1

A
1
2
3

h_5^2

B
1
3

h_4^2

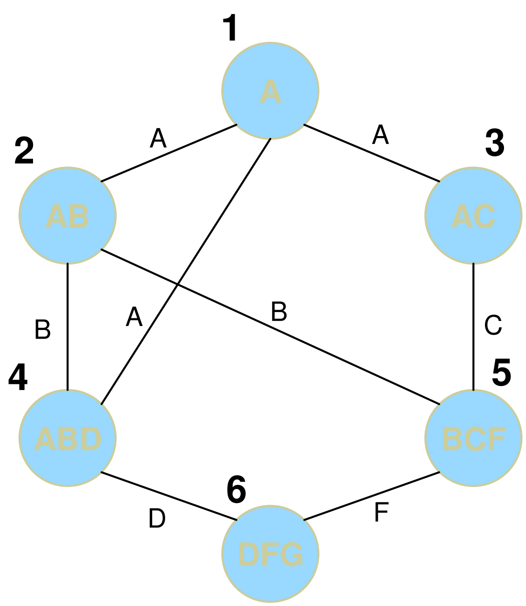
B
1
2
3

h_1^2

A
1
2
3

R_2

A	B
1	2
1	3
2	1
2	3
3	1
3	2



R_3

A	C
1	2
3	2

h_1^3

A
1
2
3

h_5^3

C
2

h_6^4

D
1
2

h_2^4

B
1
2
3

h_1^4

A
1
2
3

R_4

A	B	D
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

R_5

B	C	F
1	2	3
3	2	1

h_2^5

B
1
2
3

h_3^5

C
2

h_6^5

F
1
3

R_6

D	F	G
1	2	3
2	1	3

h_4^6

D
1
2
3

h_5^6

F
1
3

$$R_i \leftarrow R_i \cap \left(\bowtie_{k \in ne(i)} h_k^i\right) \qquad (2)$$

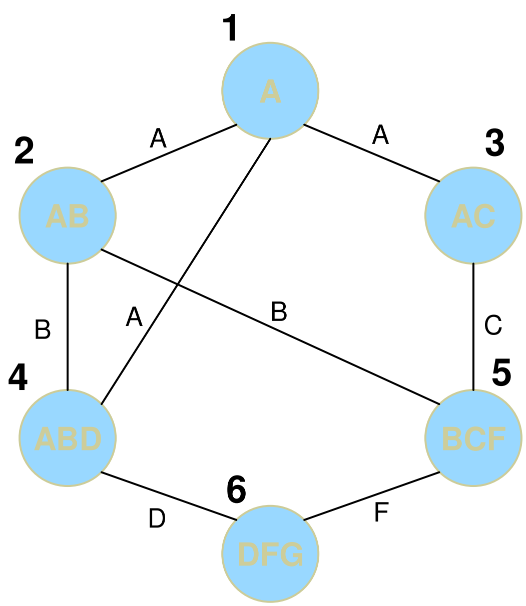
Iteration 1

R_1

A
1
3

R_2

A	B
1	3
2	1
2	3
3	1



R_3

A	C
1	2
3	2

R_4

A	B	D
1	3	2
2	3	1
3	1	2
3	2	1

R_5

B	C	F
1	2	3
3	2	1

R_6

D	F	G
2	1	3

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

Iteration 2

R_1

A
1
3

h_2^1

A
1
2
3

h_3^1

A
1
3

h_4^1

A
1
2
3

h_5^2

B
1
3

h_4^2

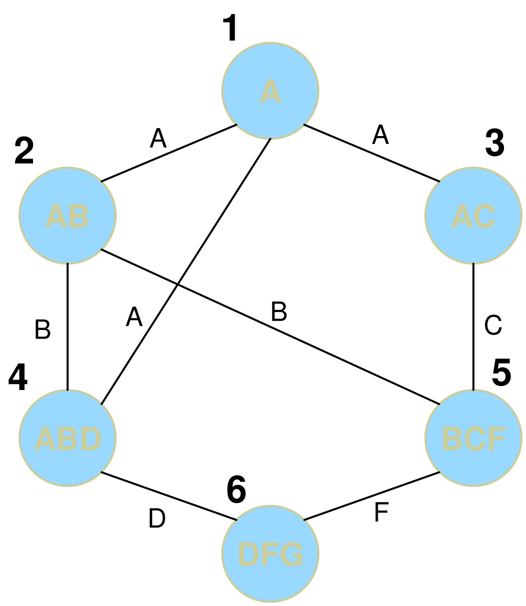
B
1
2
3

h_1^2

A
1
3

R_2

A	B
1	3
2	1
2	3
3	1



R_3

A	C
1	2
3	2

h_1^3

A
1
3

h_5^3

C
2

h_6^4

D
2

h_2^4

B
1
3

h_1^4

A
1
3

R_4

A	B	D
1	3	2
2	3	1
3	1	2
3	2	1

R_5

B	C	F
1	2	3
3	2	1

h_2^5

B
1
3

h_3^5

C
2

h_6^5

F
1

R_6

D	F	G
2	1	3

h_4^6

D
1
2

h_5^6

F
1
3

$$R_i \leftarrow R_i \cap \left(\bowtie_{k \in ne(i)} h_k^i\right) \qquad (2)$$

Iteration 2

R_1

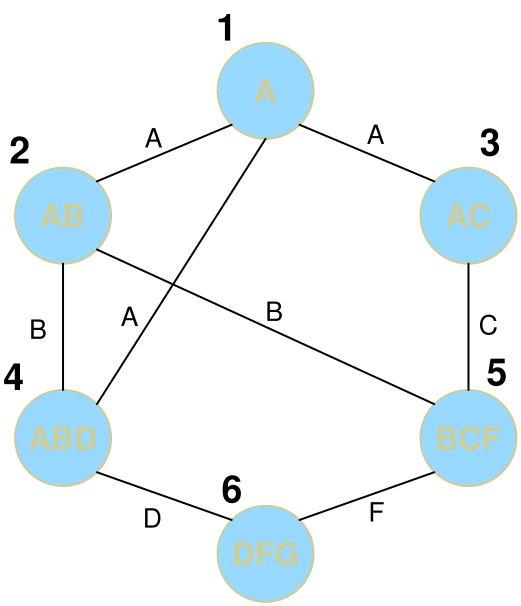
A
1
3

R_2

A	B
1	3
3	1

R_3

A	C
1	2
3	2



R_4

A	B	D
1	3	2
3	1	2

R_5

B	C	F
3	2	1

R_6

D	F	G
2	1	3

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k\in ne(i)} h_k^i)) \qquad (1)$$

Iteration 3

R_1

A
1
3

h_2^1

A
1
3

h_3^1

A
1
3

h_4^1

A
1
3

h_5^2

B
3

h_4^2

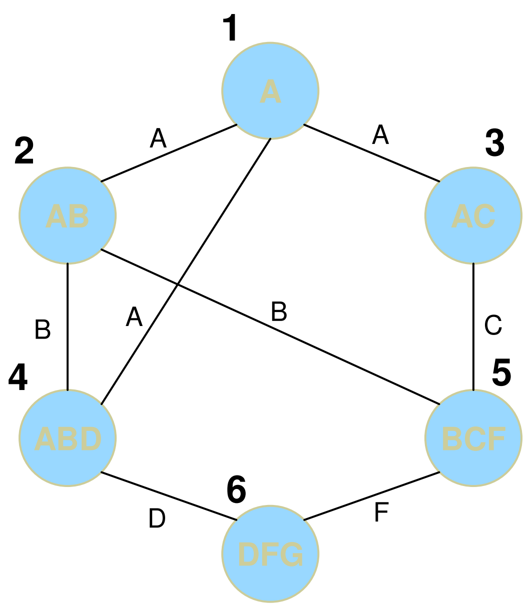
B
1
3

h_1^2

A
1
3

R_2

A	B
1	3
3	1



R_3

A	C
1	2
3	2

h_1^3

A
1
3

h_5^3

C
2

h_6^4

D
2

h_2^4

B
1
3

h_1^4

A
1
3

R_4

A	B	D
1	3	2
3	1	2

R_5

B	C	F
3	2	1

h_2^5

B
1
3

h_3^5

C
2

h_6^5

F
1

R_6

D	F	G
2	1	3

h_4^6

D
2

h_5^6

F
1

$$R_i \leftarrow R_i \cap \left(\bowtie_{k \in ne(i)} h_k^i\right) \qquad (2)$$

Iteration 3

R_1

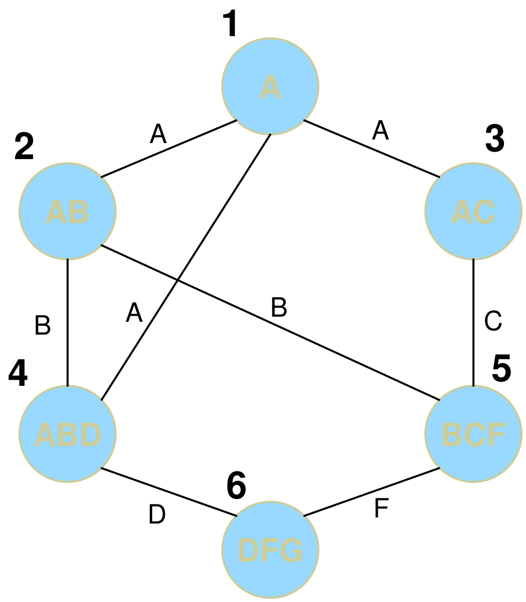
A
1
3

R_2

A	B
1	3

R_3

A	C
1	2
3	2



R_4

A	B	D
1	3	2
3	1	2

R_5

B	C	F
3	2	1

R_6

D	F	G
2	1	3

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k\in ne(i)} h_k^i)) \qquad (1)$$

Iteration 4

R_1

A
1
3

h_2^1

A
1

h_3^1

A
1
3

h_4^1

A
1
3

h_5^2

B
3

h_4^2

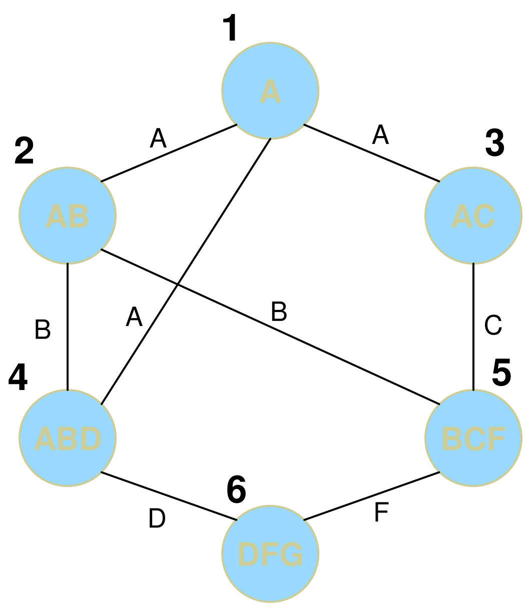
B
1
3

h_1^2

A
1
3

R_2

A	B
1	3



R_3

A	C
1	2
3	2

h_1^3

A
1
3

h_5^3

C
2

h_6^4

D
2

h_2^4

B
1
3

h_1^4

A
1
3

R_4

A	B	D
1	3	2
3	1	2

R_5

B	C	F
3	2	1

h_2^5

B
3

h_3^5

C
2

h_6^5

F
1

R_6

D	F	G
2	1	3

h_4^6

D
2

h_5^6

F
1

$$R_i \leftarrow R_i \cap \left(\bowtie_{k\in ne(i)} h_k^i\right) \tag{2}$$

Iteration 4

R_1

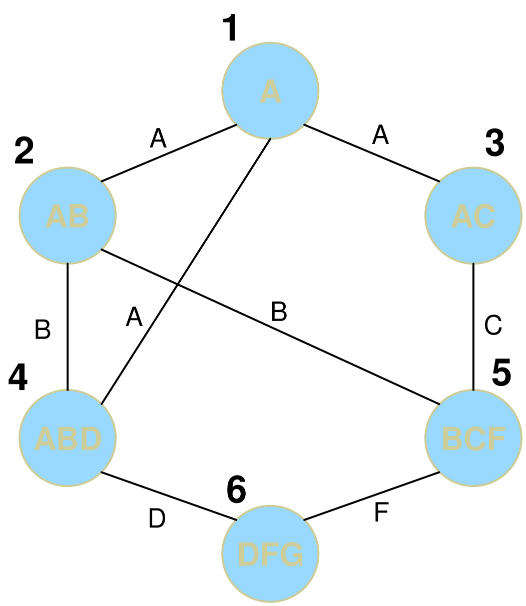
A
1

R_2

A	B
1	3

R_3

A	C
1	2
3	2



R_4

A	B	D
1	3	2

R_5

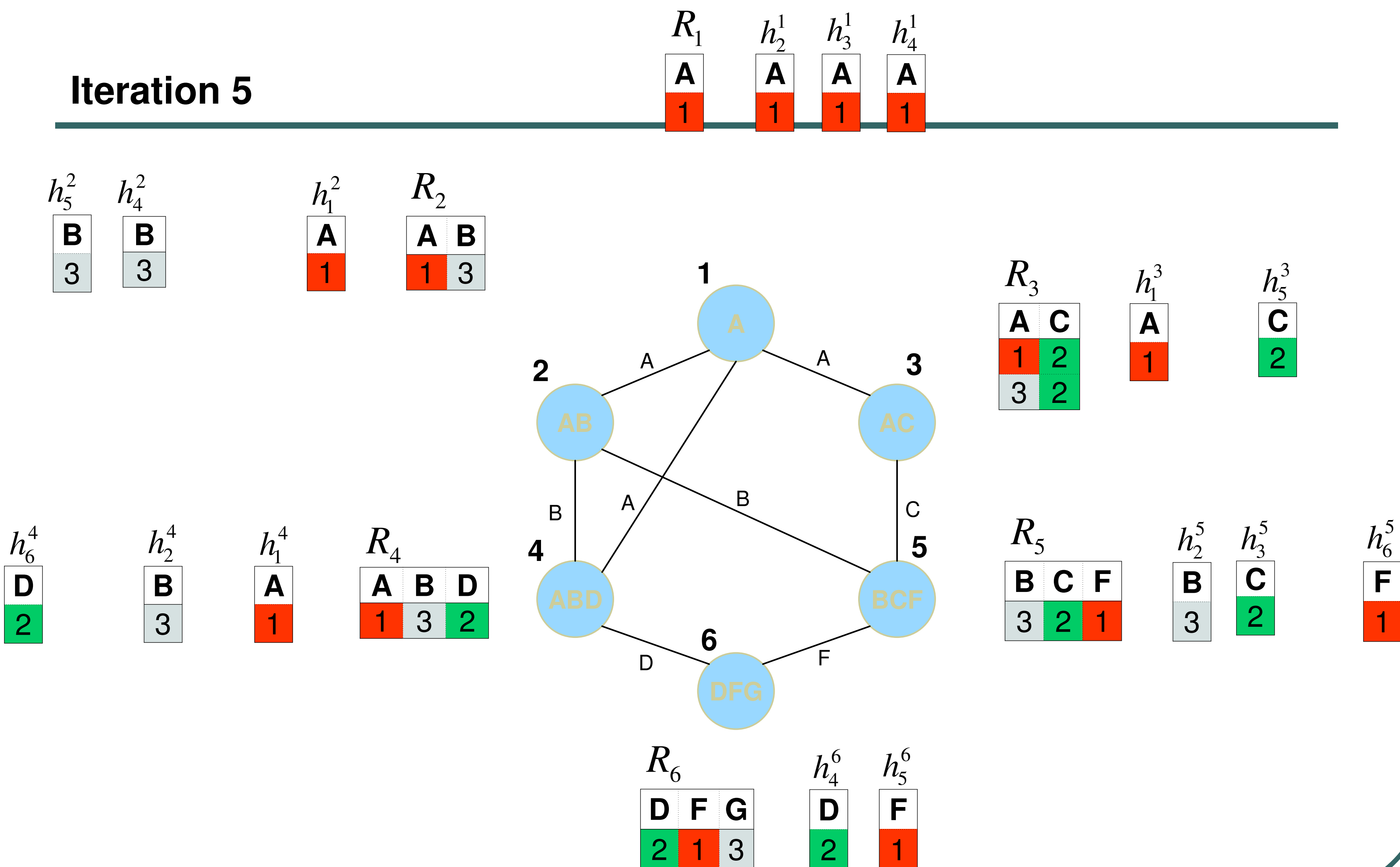
B	C	F
3	2	1

R_6

D	F	G
2	1	3

$$h_i^j \leftarrow \pi_{l_{ij}}(R_i \bowtie (\bowtie_{k \in ne(i)} h_k^i)) \qquad (1)$$

Iteration 5



$$R_i \leftarrow R_i \cap \left(\bowtie_{k \in ne(i)} h_k^i\right) \qquad (2)$$

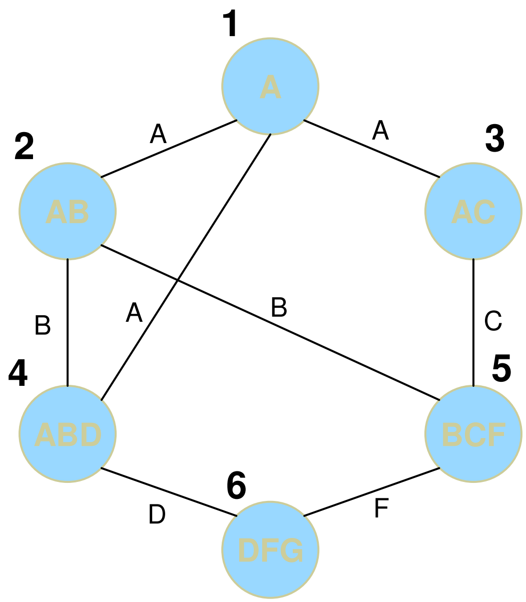
Iteration 5

R_1

A
1

R_2

A	B
1	3



R_3

A	C
1	2

R_4

A	B	D
1	3	2

R_5

B	C	F
3	2	1

R_6

D	F	G
2	1	3

Cluster Tree Elimination - properties

- Correctness and completeness: Algorithm CTE is correct, i.e. it computes the exact joint probability of every single variable and the evidence.

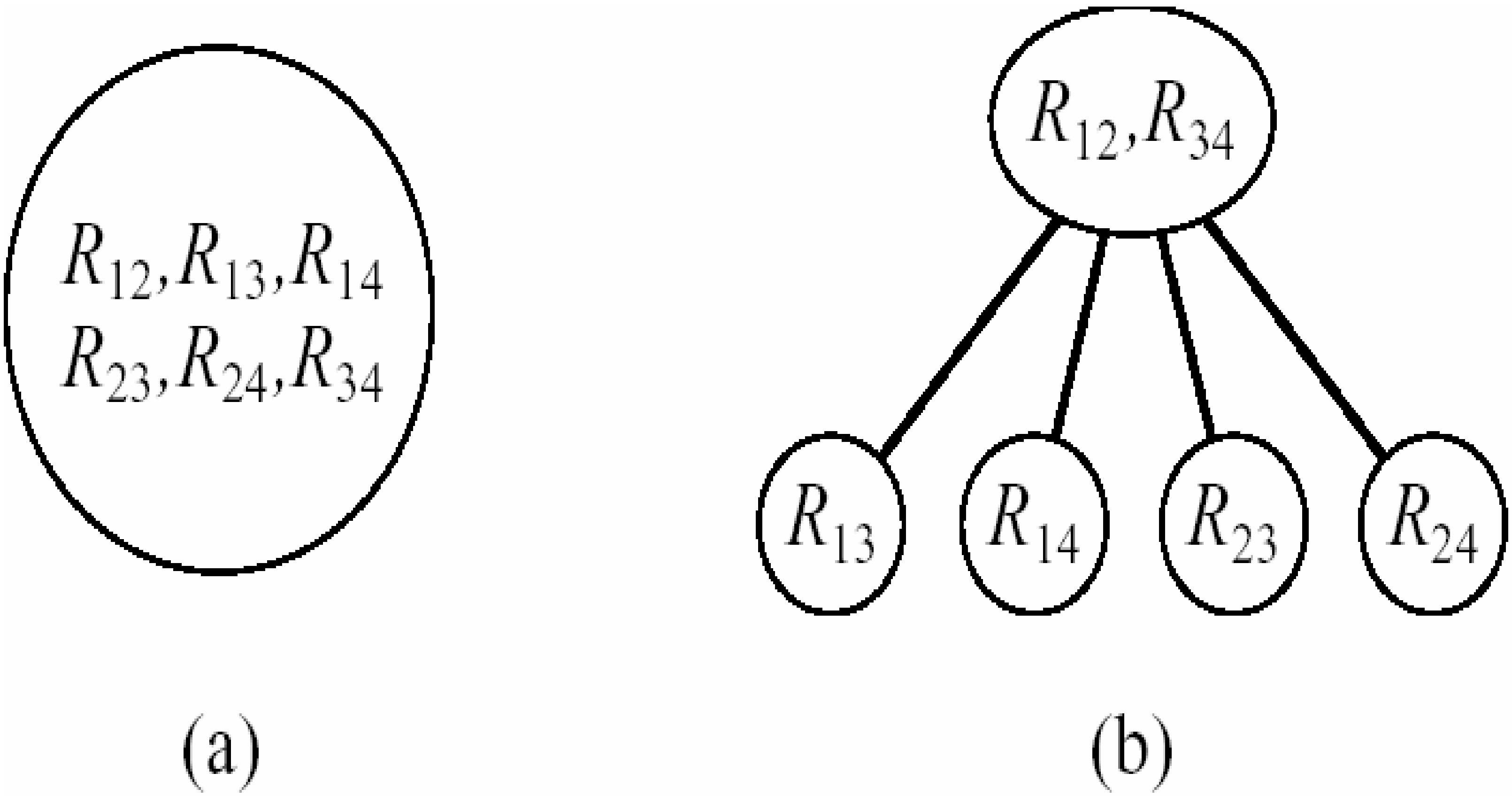
- Time complexity: $O\,(\,deg \times (n+N) \times d^{\,w^*+1}\,)$

- Space complexity: $O\,(\,N \times d^{\,sep})$

where
deg = the maximum degree of a node
n = number of variables (= number of CPTs)
N = number of nodes in the tree decomposition
d = the maximum domain size of a variable
*w** = the induced width
sep = the separator size

Time and space by hyperwidth: $O(Nt^{2hw})$, time O(N t^tw) space

Join-tree clustering is a restricted tree-decomposition

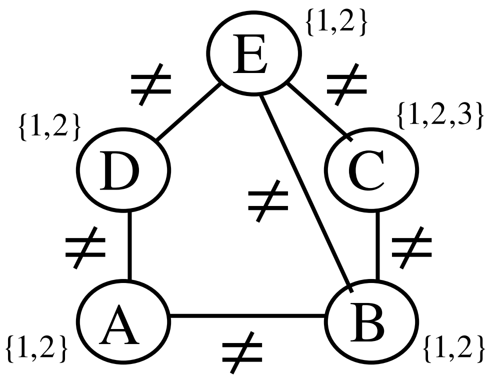


Adaptive-consistency as tree-decomposition

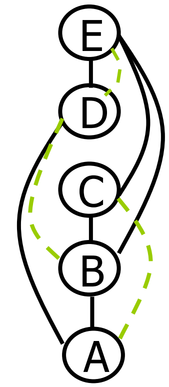
- Adaptive consistency is a message-passing along a bucket-tree
- **Bucket trees**: each bucket is a node and it is connected to a bucket to which its message is sent.
  - The variables are the clicue of the triangulated graph
  - The funcions are those placed in the initial partition

Bucket Elimination

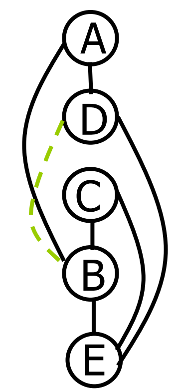
Adaptive Consistency (Dechter and Pearl, 1987)



$$
\begin{aligned}
&Bucket(E):\ \mathrm{E} \neq \mathrm{D},\ \mathrm{E} \neq \mathrm{C},\ \mathrm{E} \neq \mathrm{B}\\
&Bucket(D):\ \mathrm{D} \neq \mathrm{A}\ \ ||\ R_{DCB}\\
&Bucket(C):\ \mathrm{C} \neq \mathrm{B}\ \ ||\ R_{ACB}\\
&Bucket(B):\ \mathrm{B} \neq \mathrm{A}\ \ ||\ R_{AB}\\
&Bucket(A):\ \ R_{A}
\end{aligned}
$$



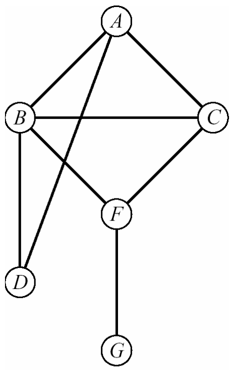
$$
\begin{aligned}
&Bucket(A):\ \mathrm{A} \neq \mathrm{D},\ \mathrm{A} \neq \mathrm{B}\\
&Bucket(D):\ \mathrm{D} \neq \mathrm{E}\ \ ||\ R_{DB}\\
&Bucket(C):\ \mathrm{C} \neq \mathrm{B},\ \mathrm{C} \neq \mathrm{E}\\
&Bucket(B):\ \mathrm{B} \neq \mathrm{E}\ \ ||\ R^{D}{}_{BE},\ R^{C}{}_{BE}\\
&Bucket(E):\ \ ||\ R_{E}
\end{aligned}
$$



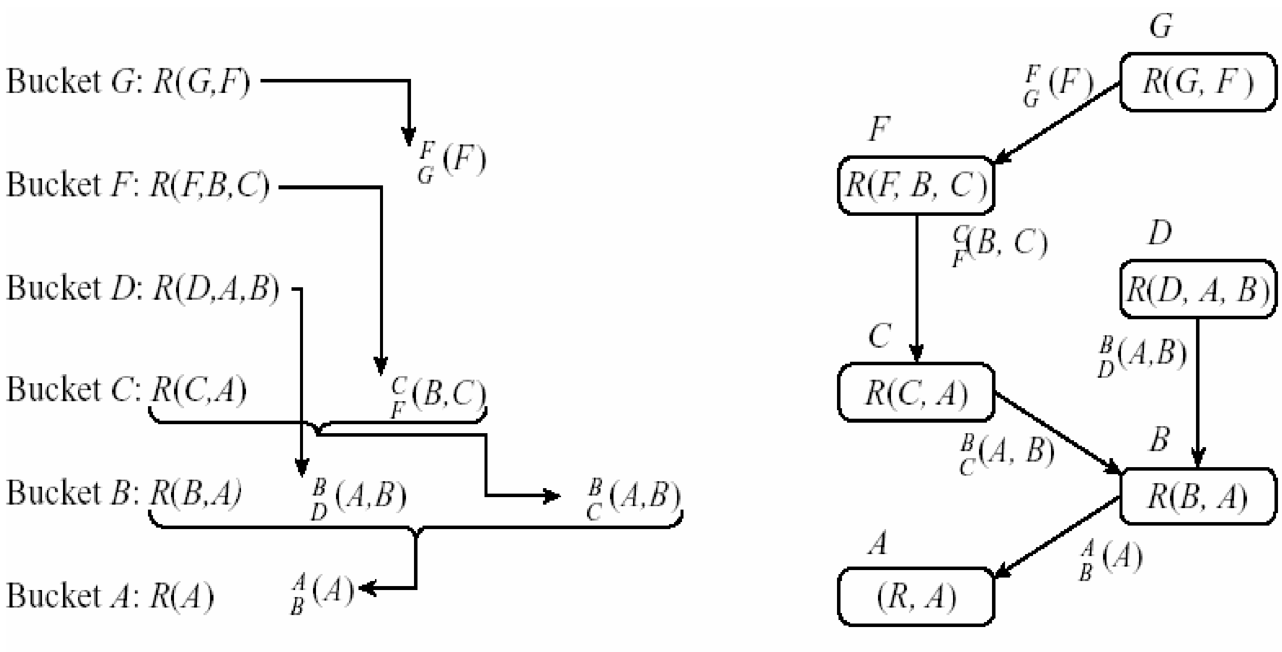
Complexity : $O(n\ exp(w^{*}(d)))$,

w^{*}(d) - ***induced width along ordering d***

From bucket-elimination to bucket-tree propagation

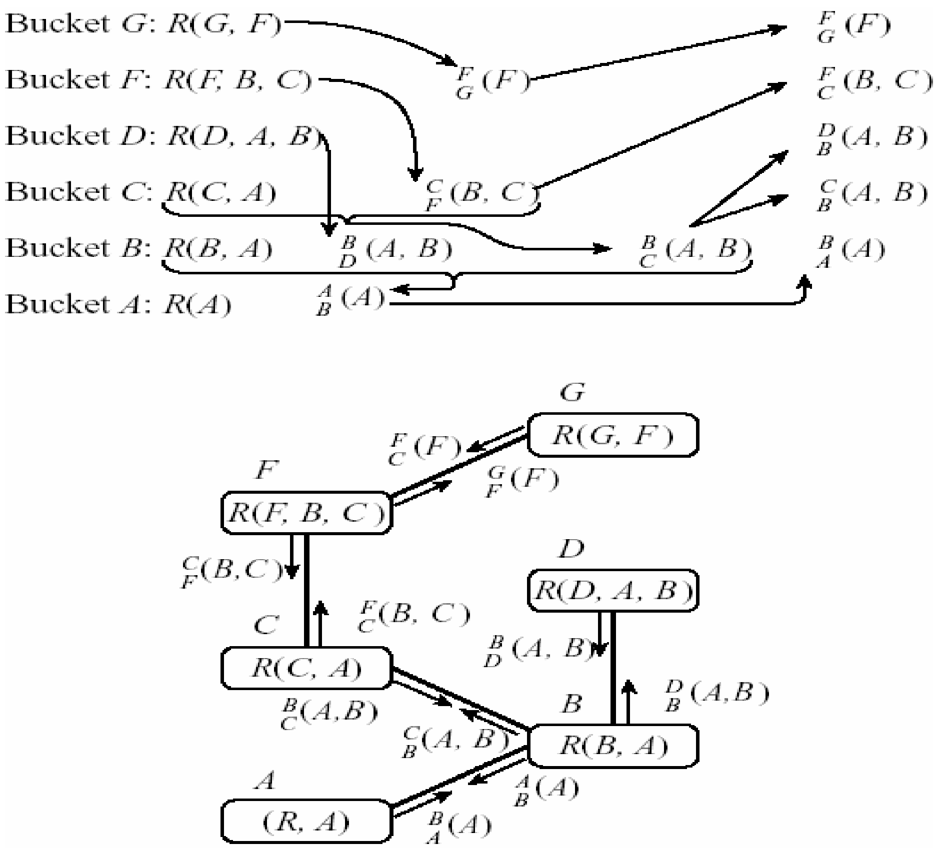


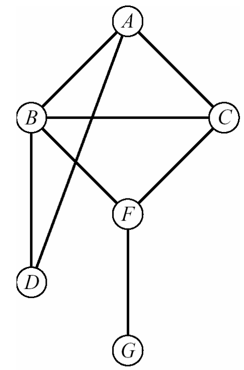
(a)



(a) (b)

The bottom up messages





(a)

Adaptive-consistency as tree-decomposition

- Adaptive consistency is a message-passing along a bucket-tree

- **Bucket trees**: each bucket is a node and it is connected to a bucket to which its message is sent.

- **Theorem**: A bucket-tree is a tree-decomposition
- Therefore, CTE adds a bottom-up message passing to bucket-elimination.