Backtracking search: look-back

Chapter 6

Look-back: backjumping

- Backjumping: Go back to the most recently culprit.
- Learning: constraint-recording, no-good recording.

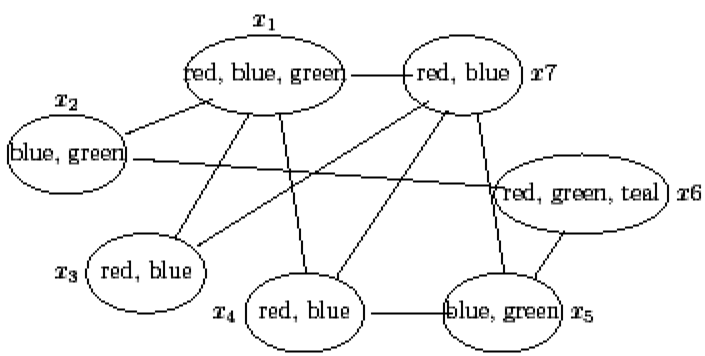
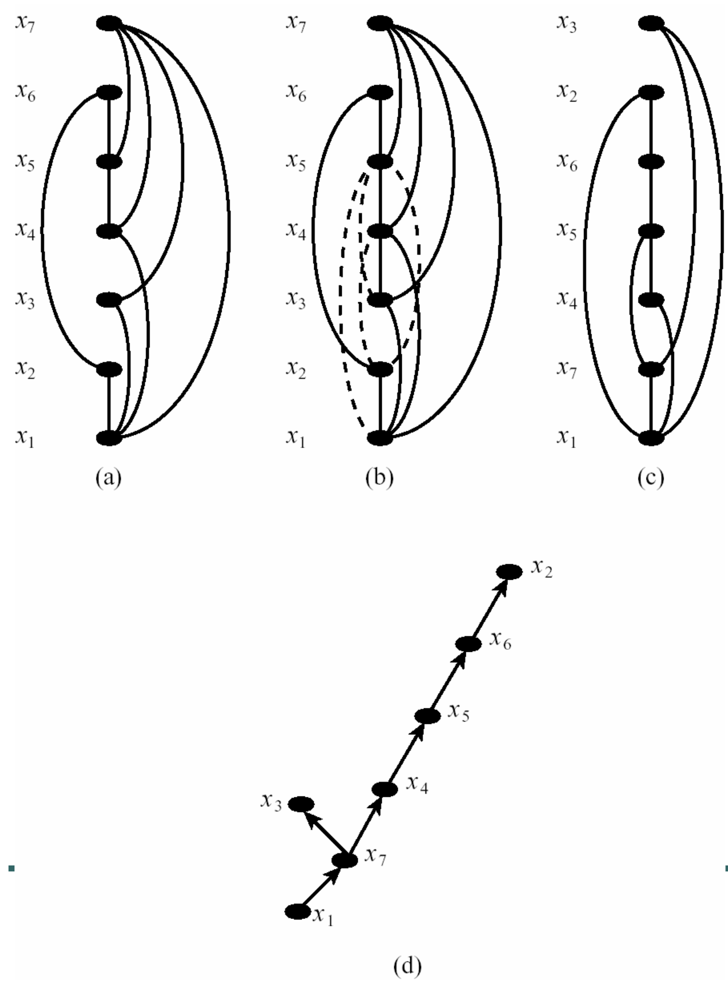
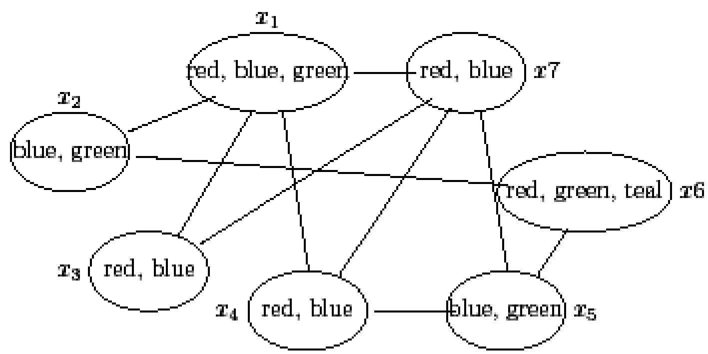


Figure 6.1: A modified coloring problem.



Backjumping, conflict sets



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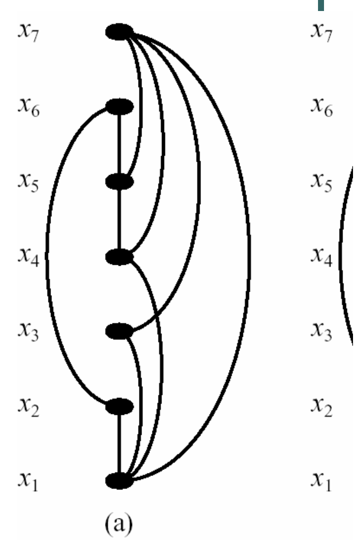


- (X1=r,x2=b,x3=b,x4=b,x5=g,x6=r,x7={r,b})
- (r,b,b,b,g,r) conflict set of x7
- (r,-,b,b,g,-) c.s. of x7
- (r,-,b,-,-,-,-) minimal c.s
- Leaf deadend: (r,b,b,b,g,r)

Example

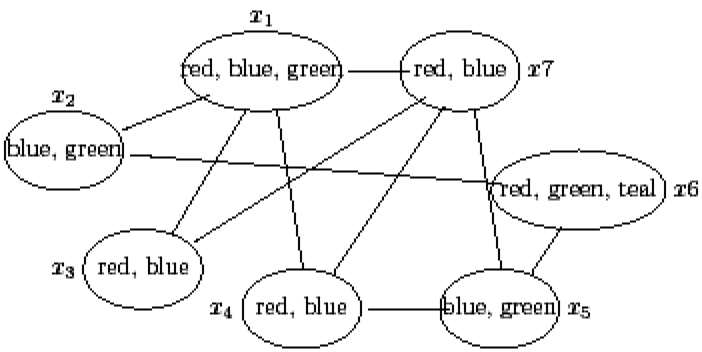
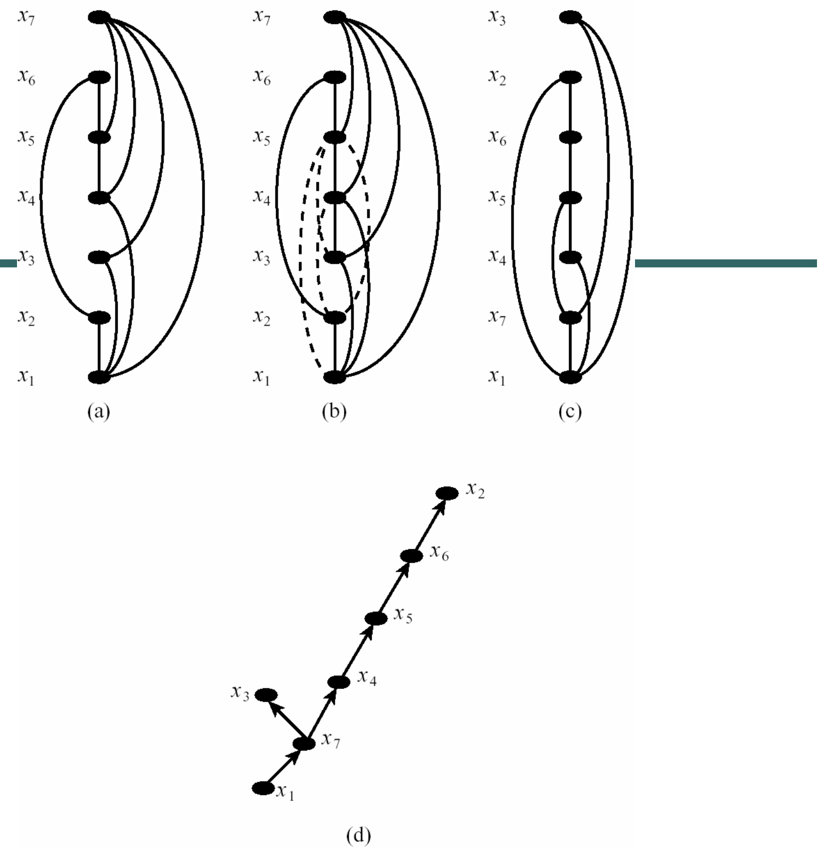


Figure 6.1: A modified coloring problem.



Example 6.1.4 For the problem in Figure 6.1, the tuple $(< x_1, red >, < x_2, blue >, < x_3, blue >, < x_4, blue >, < x_5, green >, < x_6, red >)$ is a conflict set relative to x_7 because it cannot be consistently extended to any value of x_7. It is also a leaf dead-end. Notice that the assignment $(< x_1, blue >, < x_2, green >, < x_3, red >)$ is a no-good that is not a conflict set relative to any single variable. □

Conflict-set analysis

Definition 6.1.1 (conflict set) *Let $\bar{a} = (a_{i_1}, ..., a_{i_k})$ be a consistent instantiation of an arbitrary subset of variables, and let x be a variable not yet instantiated. If there is no value b in the domain of x such that $(\bar{a}, x = b)$ is consistent, we say that $\bar{a}$ is a* conflict set *of x, or that $\bar{a}$ conflicts with variable x. If, in addition, $\bar{a}$ does not contain a subtuple that is in conflict with x, $\bar{a}$ is called a* minimal *conflict set of x.*

Definition 6.1.2 (leaf dead-end) *Let $\vec{a}_i = (a_1, ..., a_i)$ be a consistent tuple. If $\vec{a}_i$ is in conflict with x_{i+1}, it is called a* leaf dead-end.

Definition 6.1.3 (no-good) *Given a network $\mathcal{R} = (X, D, C)$, any partial instantiation $\bar{a}$ that does not appear in any solution of $\mathcal{R}$ is called a* no-good. Minimal *no-goods have no no-good subtuples.*

Definition 6.1.5 (safe jump) *Let $\vec{a}_i = (a_1, ..., a_i)$ be a leaf dead-end state. We say that x_j, where $j \leq i$, is* safe *if the partial instantiation $\vec{a}_j = (a_1, ..., a_j)$ is a no-good, namely, it cannot be extended to a solution.*

Gaschnig's backjumping:
Culprit variable

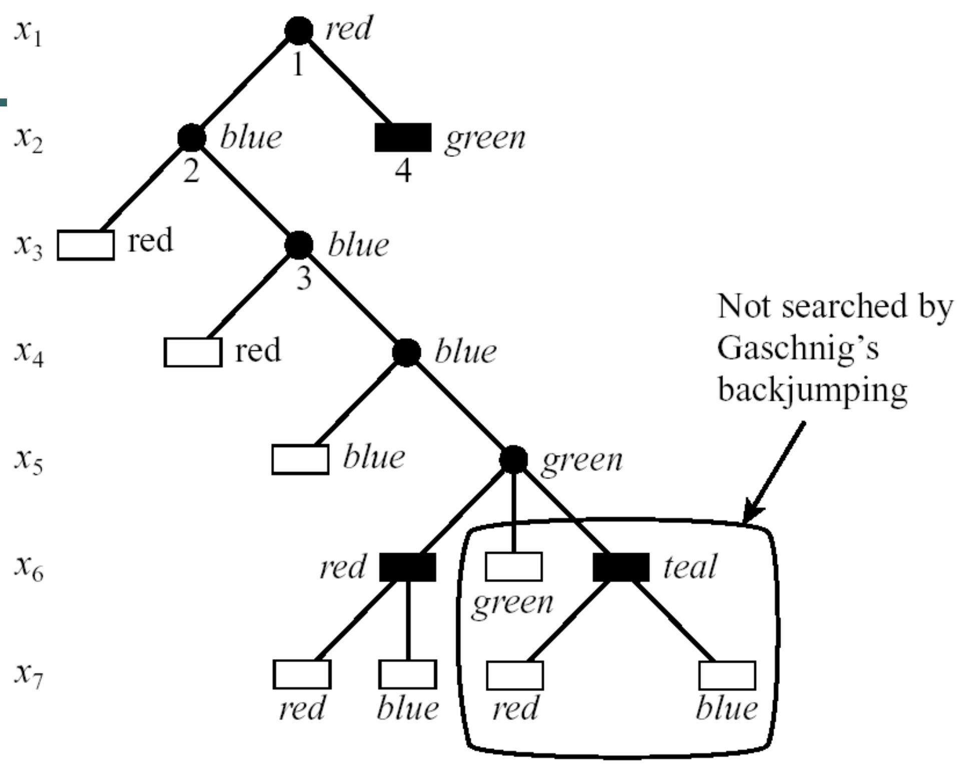
Definition 6.2.1 (culprit variable) *Let $\vec{a}_i = (a_1, ..., a_i)$ be a leaf dead-end. The* culprit index *relative to $\vec{a}_i$ is defined by $b = \min\{j \leq i |\ \vec{a}_j$ conflicts with $x_{i+1}\}$. We define the* culprit variable *of $\vec{a}_i$ to be x_b.*

- *If a_i is a leaf deadend and x_b its culprit variable, then a_b is a safe backjump destination and a_j, j<b is not.*
- *The culprit of x7 (r,b,b,b,g,r) is (r,b,b) → x3*

Gaschnig's backjumping [1979]

- Gaschnig uses a marking technique to compute the culprit.
- Each variable xj maintains a pointer (latset_j) to the latest ancestor incompatible with any of its values.
- While forward generating $\vec{a}_i$, keep array latest_i, 1<=j<-n, of pointers to the last value conflicted with some value of x_j
- The algorithm jumps from a leaf-dead-end x_{i+1} back to latest_(i+1) which is its culprit.

Example of Gaschnig's backjump



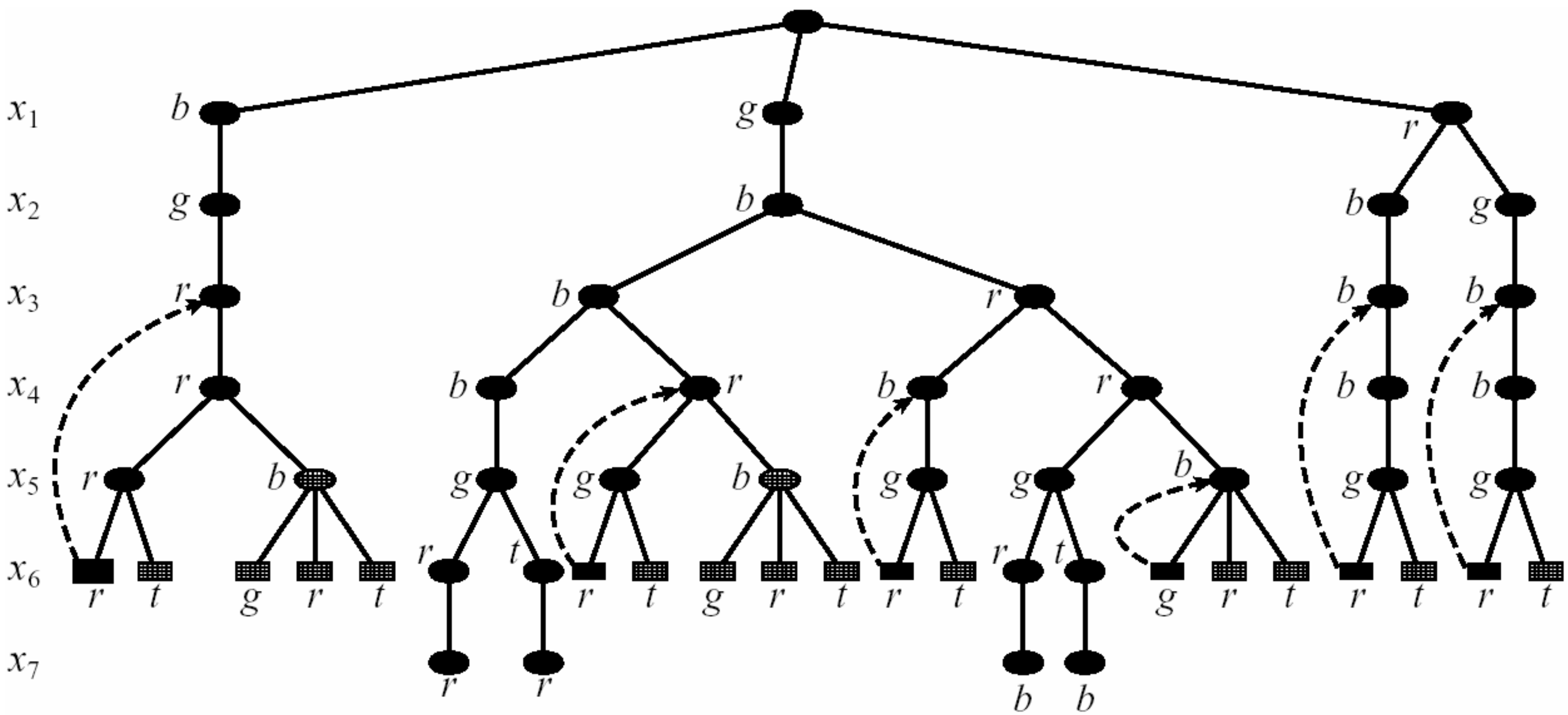
Example 6.2.3 Consider the problem in Figure 6.1 and the order d_1. At the dead-end for x_7 that results from the partial instantiation $(< x_1, red >, < x_2, blue >, < x_3, blue >, < x_4, blue >, < x_5, green >, < x_6, red >)$, $latest_7 = 3$, because $x_7 = red$ was ruled out by $< x_1, red >$, $x_7 = blue$ was ruled out by $< x_3, blue >$, and no later variable had to be examined. On returning to x_3, the algorithm finds no further values to try ($D'_3 = \emptyset$). Since $latest_3 = 2$, the next variable examined will be x_2. Thus we see the algorithm's ability to backjump at leaf dead-ends. On subsequent dead-ends, as in x_3, it goes back to its preceding variable only. An example of the algorithm's practice of pruning the search space is given in Figure 6.2. □

Properties

- Gaschnig's backjumping implements only safe and maximal backjumps in leaf-deadends.

Gaschnig jumps only at leaf-dead-ends

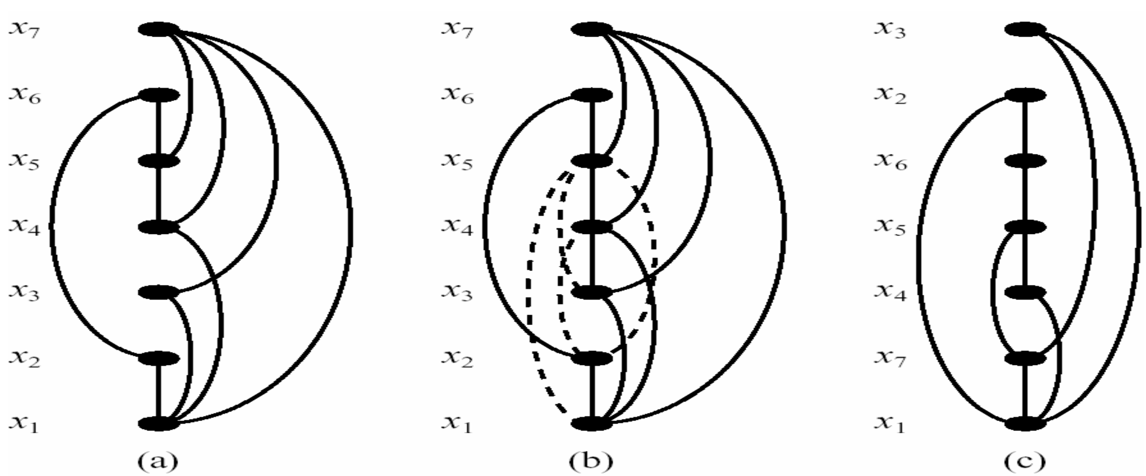
Internal dead-ends: dead-ends that are non-leaf

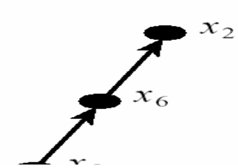


Example 6.3.1 In Figure 6.4, all of the backjumps illustrated lead to internal dead-ends, except for the jump back to $(\langle x_1, \mathit{green}\rangle, \langle x_2, \mathit{blue}\rangle, \langle x_3, \mathit{red}\rangle, \langle x_4, \mathit{blue}\rangle)$, because this is the only case where another value exists in the domain of the culprit variable. □

Example of graph-based backjumping scenarios

- Scenario 1, deadend at x4: $I_4(x_4) = \{x_1\}$
- Scenario 2: deadend at x5: $I_4(x_4, x_5) = \{x_1\}$
- Scenario 3: deadend at x7: $I_4(x_7, x_5, x_4) = \{x_1, x_3\}$
- Scenario 4: deadend at x6: $I_4(x_6, x_5, x_4) = \{x_1, x_3\}$



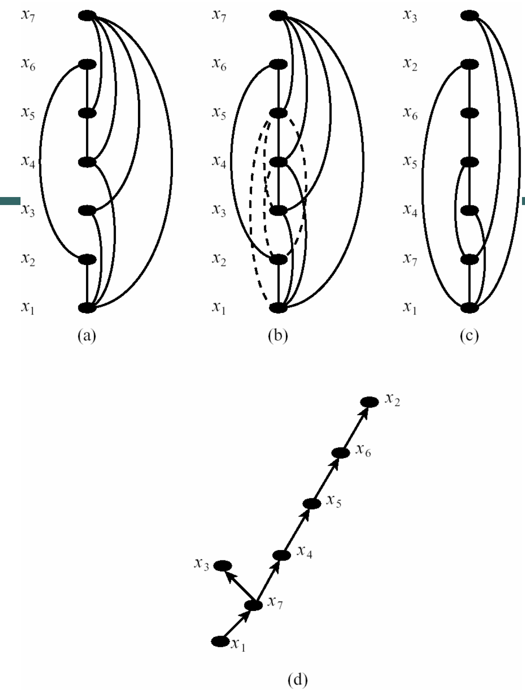


Graph-based backjumping

- Uses only graph information to find culprit
- Jumps both at leaf and at internal dead-ends
- Whenever a deadend occurs at x, it jumps to the most recent variable y connected to x in the graph. If y is an internal deadend it jumps back further to the most recent variable connected to x or y.
- The analysis of conflict is approximated by the graph.
- Graph-based algorithm provide graph-theoretic bounds.

Ancestors and parents

- anc(x7) = {x5,x3,x4,x1}
- p(x7) =x5
- p(r,b,b,b,g,r) = x5



Definition 6.3.2 (ancestors, parent) *Given a constraint graph and an ordering of the nodes* d, *the* ancestor set *of variable* x, *denoted* $anc(x)$, *is the subset of the variables that precede and are connected to* x. *The* parent *of* x, *denoted* $p(x)$, *is the most recent (or latest) variable in* $anc(x)$. *If* $\vec{a_i} = (a_1, ..., a_i)$ *is a leaf dead-end, we equate* $anc(\vec{a_i})$ *with* $anc(x_{i+1})$, *and* $p(\vec{a_i})$ *with* $p(x_{i+1})$.

Internal deadends analysis

Definition 6.3.5 (session) *We say that backtracking* invisits x_i *if it processes* x_i *coming from a variable earlier in the ordering. The session of* x_i *starts upon the invisiting of* x_i *and ends when retracting to a variable that precedes* x_i*. At a given state of the search where variable* x_i *is already instantiated, the* current session *of* x_i *is the set of variables processed by the algorithm since the most recent* invisit *to* x_i*. The current session of* x_i *includes* x_i *and therefore the session of a leaf dead-end variable has a single variable.*

Definition 6.3.6 (relevant dead-ends) *The relevant dead-ends of* x_i*'s session are defined recursively as follows. The relevant dead-ends of a leaf dead-end* x_i*, denoted* $r(x_i)$*, is* x_i*. If* x_i *is variable to which the algorithm retracted from* x_j*, then the relevant-dead-ends of* x_i *are the union of its current relevant dead-ends and the ones inherited from* x_j*, namely,* $r(x_i) = r(x_i) \cup r(x_j)$.

Definition 6.3.7 (induced ancestors, induced parent) *Let* x_i *be a variable that is an internal or leaf dead-end. Let* Y *be a subset of the variables consisting of all its relevant dead-ends in the current session of* x_i*. We denote* $anc(Y) = \cup_{y \in Y} anc(y)$*. The* induced ancestor set *of* x_i *relative to* Y*,* $I_i(Y)$*, is the union of all* Y*'s ancestors, restricted to variables that precede* x_i*. Formally,* $I_i(Y) = anc(Y) \cap \{x_1, ..., x_{i-1}\}$*. The induced parent of* x_i *relative to* Y*,* $P_i(Y)$*, is the latest variable in* $I_i(Y)$*. We call* $P_i(Y)$ *the graph-based culpribt of* x_i*.*

Graph-based back-jumping algorithm, but we need to jump at internal dead-ends too

procedure GRAPH-BASED-BACKJUMPING
Input: A constraint network $\mathcal{R} = (X, D, C)$
Output: Either a solution, or a decision that the network is inconsistent.

compute $anc(x_i)$ for each x_i (see Definition 6.3.2 in text)
$i \leftarrow 1$ (initialize variable counter)
$D'_i \leftarrow D_i$ (copy domain)
$I_i \leftarrow anc(x_i)$ (copy of anc() that can change)
while $1 \le i \le n$
  instantiate $x_i \leftarrow$ SELECTVALUE
  if x_i is null (no value was returned)
    $iprev \leftarrow i$
    $i \leftarrow$latest index in I_i (backjump)
    $I_i \leftarrow I_i \cup I_{iprev} - \{x_i\}$
  else
    $i \leftarrow i + 1$
    $D'_i \leftarrow D_i$
    $I_i \leftarrow anc(x_i)$
end while
if $i = 0$
  return "inconsistent"
else
  return instantiated values of $\{x_1, \ldots, x_n\}$
end procedure

procedure SELECTVALUE (same as BACKTRACKING's)
while D'_i is not empty
  select an arbitrary element $a \in D'_i$, and remove a from D'_i
  if CONSISTENT$(\vec{a}_{i-1}, x_i = a)$
    return a
end while
return null (no consistent value)
end procedure

Figure 6.5: The graph-based backjumping algorithm.

Properties of graph-based back-jumping

- Algorithm graph-based back-jumping jumps back at any dead-end variable as far as graph-based information allows.

- For each variable, the algorithm maintains the induced-ancestor set I_i relative the relevant dead-ends in its current session.

Conflict-directed backjumping
(Prosser 1990)

- Extend Gaschnig's backjump to internal dead-ends.
- Exploits information gathered during search.
- For each variable the algorithm maintains an induced jumpback set, and jumps to most recent one.
- **Use the following concepts:**
  - An ordering over variables induced a strict ordering between constraints**:** R1<R2<…Rt
  - Use **earliest minimal consflict-set (**emc(x_(i+1)) **)** of a deadend.
  - Define the **jumpback set** of a deadend

Conflict-directed backjumping: Gaschnig's style jumpback in all deadends:

Definition 6.4.1 (earlier constraint) *Given an ordering of the variables in a constraint problem, we say that constraint R is* earlier *than constraint Q if the latest variable in $scope(R) - scope(Q)$ precedes the latest variable in $scope(Q) - scope(R)$.*

Definition 6.4.2 (earliest minimal conflict set) *For a network $\mathcal{R} = (X, D, C)$ with an ordering of the variables d, let $\vec{a}_i$ be a leaf dead-end tuple whose dead-end variable is x_{i+1}. The* earliest minimal conflict set *of $\vec{a}_i$, denoted $emc(\vec{a}_i)$, can be generated as follows. Consider the constraints in $C = \{R_1, \ldots, R_c\}$ with scopes $\{S_1, \ldots, S_c\}$, in order as defined in Definition 6.4.1. For $j = 1$ to c, if there exists $b \in D_{i+1}$ such that R_j is violated by $(\vec{a}_i, x_{i+1} = b)$, but no constraint earlier than R_j is violated by $(\vec{a}_i, x_{i+1} = b)$, then var-$emc(\vec{a}_i) \leftarrow$ var-$emc(\vec{a}_i) \cup S_j$. $emc(\vec{a}_i)$ is the subtuple of $\vec{a}_i$ containing just the variable-value pairs of var-$emc(\vec{a}_i)$. Namely, $emc(\vec{a}_i) = \vec{a}_i[var - emc(\vec{a}_i)]$.*

Definition 6.4.3 (jumpback set) *The* jumpback set *of a leaf dead-end J_{i+1} of x_{i+1} is its var-$emc(\vec{a}_i)$. The jump-back set of an internal state $\vec{a}_i$ includes all the var-$emc(\vec{a}_j)$ of all the relevant dead-ends $\vec{a}_j$ $j \geq i$, that occurred in the current session of x_i. Formally, $J_i = \bigcup\{var\text{-}emc(\vec{a}_j) \mid \vec{a}_j \text{ is a relevant dead-end in } x_i\text{'s session}\}$*

Example of conflict-directed backjumping

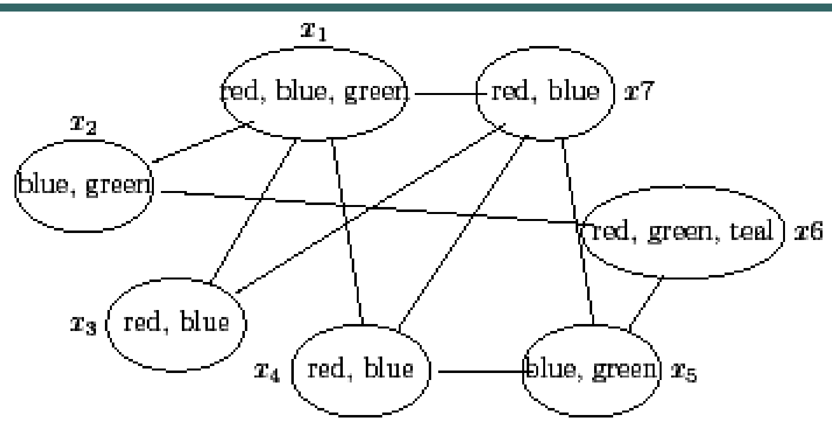


Figure 6.1: A modified coloring problem.

Example 6.4.5 Consider the problem of Figure 6.1 using ordering $d_1 = (x_1, \ldots, x_7)$. Given the dead-end at x_7 and the assignment $\vec{a_6} = (blue, green, red, red, blue, red)$, the emc set is $(< x_1, blue >, < x_3, red >)$, since it accounts for eliminating all the values of x_7. Therefore, algorithm conflict-directed backjumping jumps to x_3. Since x_3 is an internal dead-end whose own $var - emc$ set is $\{x_1\}$, the jumpback set of x_3 includes just x_1, and the algorithm jumps again, this time back to x_1. $\square$

Properties

- Given a dead-end $\vec{a}_i$, the latest variable in its jumpback set J_i is the earliest variable to which it is safe to jump.
- This is the culprit.
- Algorithm conflict-directed backtracking jums back to the latest variable in the dead-ends's jumpback set, and is therefore safe and maximal.

Conflict-directed backjumping

procedure CONFLICT-DIRECTED-BACKJUMPING
Input: A constraint network $\mathcal{R} = (X, D, C)$.
Output: Either a solution, or a decision that the network is inconsistent.

```
    i ← 1                                  (initialize variable counter)
    D'_i ← D_i                             (copy domain)
    J_i ← ∅                                (initialize conflict set)
    while 1 ≤ i ≤ n
       instantiate x_i ← SELECTVALUE-CBJ
       if x_i is null                      (no value was returned)
          iprev ← i
          i ← index of last variable in J_i    (backjump)
          J_i ← J_i ∪ J_iprev − {x_i}(merge conflict sets)
       else
          i ← i + 1                        (step forward)
          D'_i ← D_i                       (reset mutable domain)
          J_i ← ∅                          (reset conflict set)
    end while
    if i = 0
       return "inconsistent"
    else
       return instantiated values of {x_1, ... , x_n}
end procedure

subprocedure SELECTVALUE-CBJ

    while D'_i is not empty
       select an arbitrary element a ∈ D'_i, and remove a from D'_i
       consistent ← true
       k ← 1
       while k < i and consistent
          if CONSISTENT(a⃗_k, x_i = a)
             k ← k + 1
          else
             let R_S be the earliest constraint causing the conflict
             add the variables in R_S's scope S, but not x_i, to J_i
             consistent ← false
       end while
       if consistent
          return a
    end while
    return null                            (no consistent value)
end procedure
```

Figure 6.7: The conflict-directed backjumping algorithm.

Graph-based backjumping on DFS orderings

Example 6.5.1 Consider, once again, the CSP in Figure 6.1. A *DFS* ordering $d_2 = (x_1, x_7, x_4, x_5, x_6, x_2, x_3)$ and its corresponding *DFS* spanning tree are given in Figure 6.6c,d. If a dead-end occurs at node x_3, the algorithm retreats to its *DFS* parent, which is x_7. □

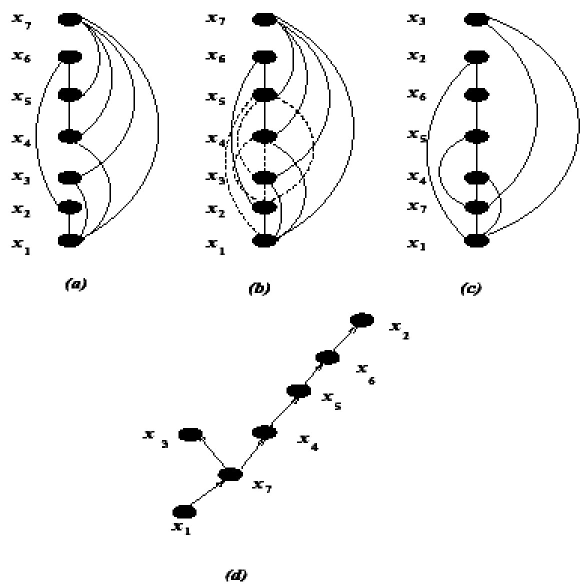


Figure 6.6: Several ordered constraint graphs of the problem in Figure 6.1: (a) along ordering $d_1 = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$, (b) the induced graph along d_1, (c) along ordering $d_2 = (x_1, x_7, x_4, x_5, x_6, x_2, x_3)$, and (d) a DFS spanning tree along ordering d_2.

Graph-based backjumping on DFS ordering

- Example:d = x1,x2,x3,x4,x5,x6,x7
- Constraints: (6,7)(5,2)(2,3)(5,7)(2,7)(2,1)(2,3)(1,4)3,4)
- Rule: go back to parent. No need to maintain parent set

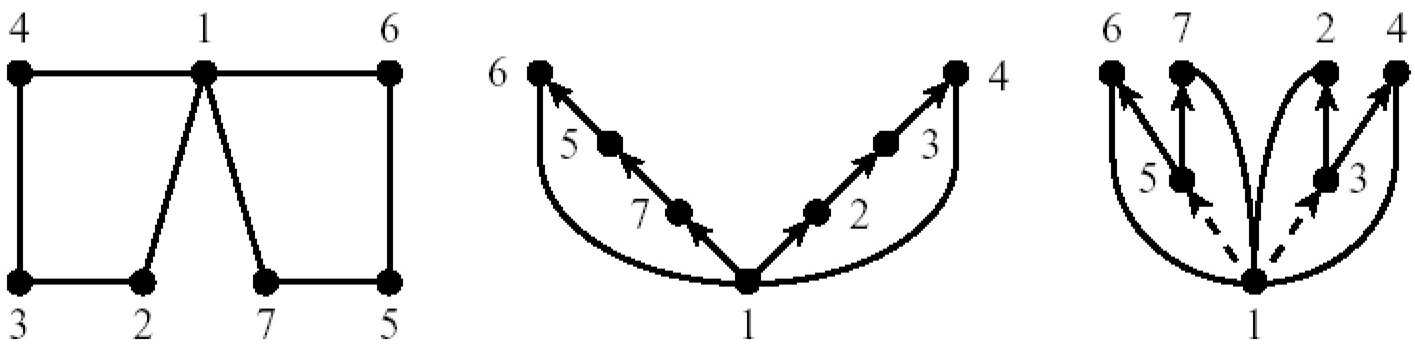
Theorem 6.5.2 *Given a DFS ordering of the constraint graph, if $f(x)$ denotes the* DFS *parent of x, then, upon a dead-end at x, $f(x)$ is x's graph-based earliest safe variable for both leaf and internal dead-ends.*

Complexity of graph-based backjumping on DFS ordering

- T_i= number of nodes in the And-Or search space rooted at x_i (level m-i)
- Each assignment of a value to x_i generates subproblems:
  - T_i = k b T_{i-1}
  - T_0 = k
- Solution: $T_m = b^m k^{m+1}$

Theorem 6.5.3 *When graph-based backjumping is performed on a* DFS *ordering of the constraint graph, the number of nodes visited is bounded by $O((b^m k^{m+1}))$, where b bounds the branching degree of the* DFS *tree associated with that ordering, m is its depth and k is the domain size. The time complexity (measured by the number of consistency checks) is $O(ek(bk)^m)$, where e is the number of constraints.*

DFS of induced graphs



(a) (b) (c)

Theorem 6.5.5 *If d is a* DFS *ordering of* (G^*, d_1) *for some ordering d_1, having depth m^*_d, then the complexity of graph-based backjumping using ordering d is* $O(\exp(m^*_d))$.

Graph parameters

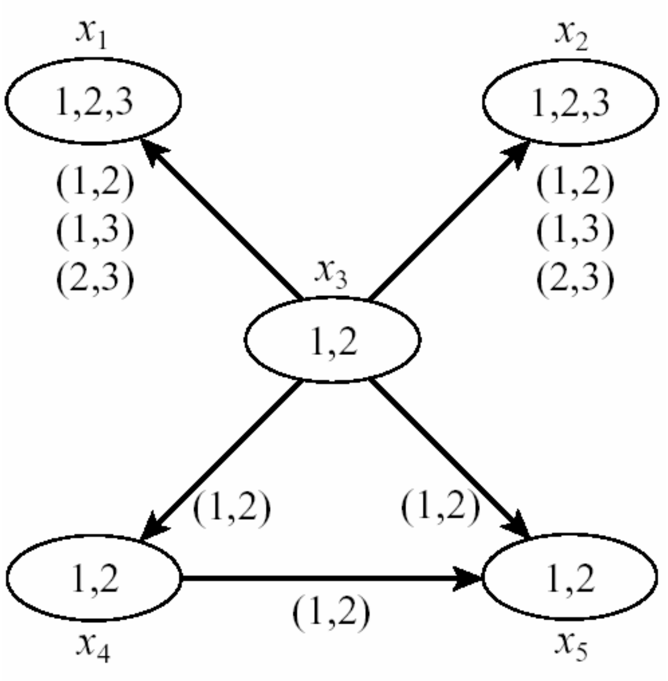
- C- size of a cycle-cutset
- m- depth of a dfs in any induced graph
- m_s a simple depth of a dfs tree.

- What is the relationship between these?

Learning, constraint recording

- Learning means recording conflict sets
- An opportunity to learn is when deadend is discovered.
- Goal of learning to not discover the same deadends.
- Try to identify small conflict sets
- Learning prunes the search space.

Look-back: constraint recording



- (x1=2,x2=2,x3=1,x4=2) IS a dead-end
- Conflicts to record:
- (x1=2,x2=2,x3=1,x4=2) 4-ary
- (x3=1,x4=2) binary
- (x4=2) unary

Learning algorithms

- Graph-based learning
- Deep vs shallow learning
- Jumpback learning
- Non-systematic randomized learning
- Complexity of backtracking with learning
- Look-back for SAT

Learning example

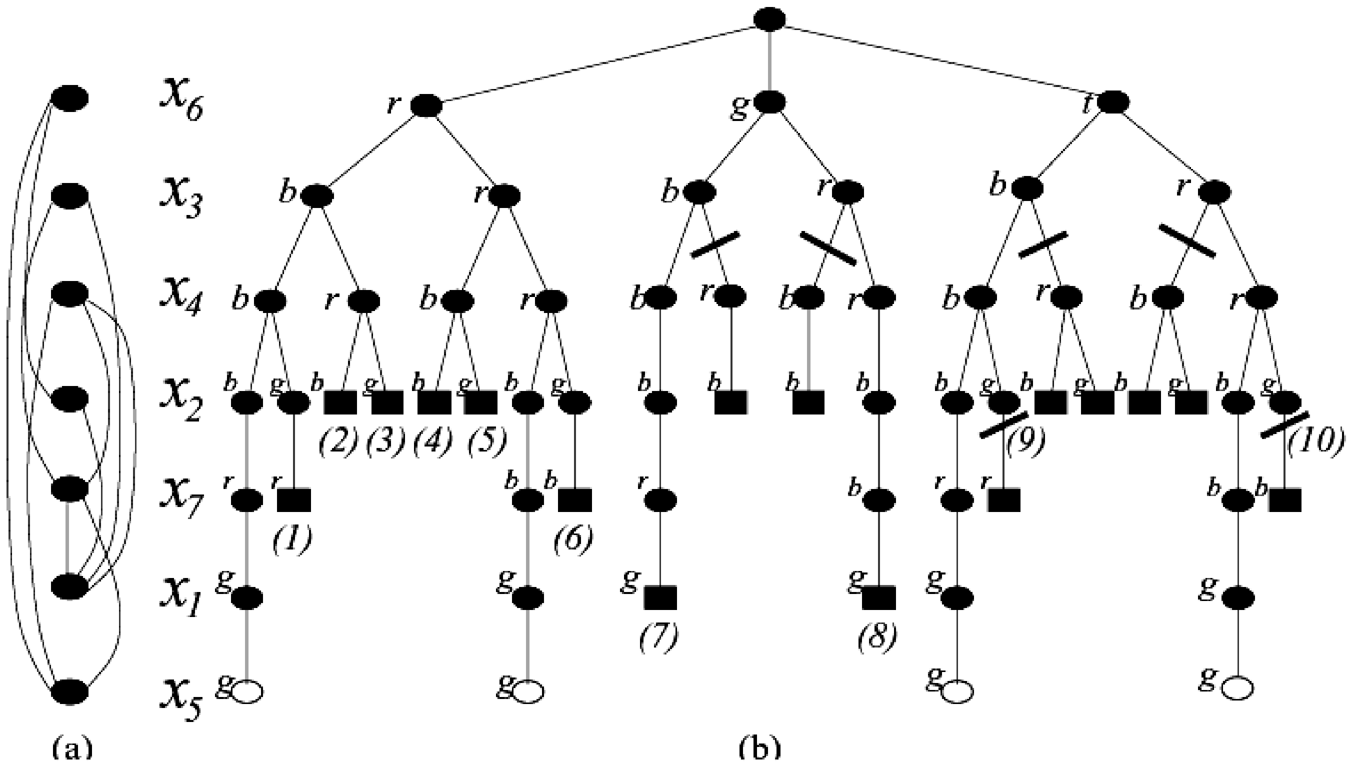


Figure 6.9: The search space explicated by backtracking on the CSP from Figure 6.1, using the variable ordering $(x_6, x_3, x_4, x_2, x_7, x_1, x_5)$ and the value ordering (*blue, red, green, teal*). Part (a) shows the ordered constraint graph, part (b) illustrates the search space. The cut lines in (b) indicate branches not explored when graph-based learning is used.

Graph-based learning algorithm

procedure GRAPH-BASED-BACKJUMP-LEARNING

instantiate $x_i \leftarrow$ SELECTVALUE
if x_i is null (no value was returned)
record a constraint prohibiting $\vec{a}_{i-1}[I_i]$**.**
$iprev \leftarrow i$
(algorithm continues as in Fig. 6.5)

Figure 6.10: Graph-based backjumping learning, modifying CBJ

Deep learning

- Deep learning: recording all and only minimal conflict sets
- Example:
- Although most accurate, overhead is prohibitive: the number of conflict sets in the worst-case:

$$\binom{r}{r/2} = 2^r$$

Jumpback learning

- Record the jumpback assignment

Example 6.7.2 For the problem and ordering of Example 6.7.1 at the first dead-end, jumpback learning will record the no-good ($x_2 = green$, $x_3 = blue$, $x_7 = red$), since that tuple includes the variables in the jumpback set of x_1. □

procedure CONFLICT-DIRECTED-BACKJUMP-LEARNING

instantiate $x_i \leftarrow$ SELECTVALUE-CBJ
if x_i is null (no value was returned)
  record a constraint prohibiting $\vec{a}_{i-1}[J_i]$ **and corresponding values**
  $iprev \leftarrow i$
  (algorithm continues as in Fig. 6.7)

Figure 6.11: Conflict-directed bakjump-learning, modifying CBJ

Bounded and relevance-based learning

Bounding the arity of constraints recorded.

- When bound is i: i-ordered graph-based,i-order jumpback or i-order deep learning.
- Overhead complexity of i-bounded learning is time and space exponential in i.

Definition 6.7.3 (i-relevant) *A no-good is i-relevant if it differs from the current partial assignment by at most i variable-value pairs.*

Definition 6.7.4 (i'th order relevance-bounded learning) *An i'th order relevance-bounded learning scheme maintains only those learned no-goods that are i-relevant.*

Non-systematic randomized learning

- Do search in a random way with interrupts, restarts, unsafe backjumping, **but record conflicts**.
- Guaranteed completeness.

Complexity of backtrack-learning

Theorem 6.7.5 *Let d be an ordering of a constraint graph, and let $w^*(d)$ be its induced width. Any backtracking algorithm using ordering d with graph-based learning has a space complexity of $O((nk)^{w^*(d)+1})$ and a time complexity of $O((2nk)^{w^*(d)+1})$, where n is the number of variables and k bounds the domain sizes.*

The number of dead-ends is bounded by the number of possible no-goods of size w*

$$\sum_{i=1}^{w^*(d)} \binom{n}{i} k^i = O((nk)^{w^*(d)+1})$$

Number of constraint tests per dead-end are

$$O(2^{w^*(d)})$$

Complexity of backtrack-learning (refined)

- **Theorem:** Any backtracking algorithm using graph-based learning along d has a space complexity O(n k^w*(d)) and time complexity O(n^2 (2k)^(w*(d)+1) (book). Refined more: O(n^2 k^w*(d))

- **Proof:** The number of deadends for each variable is O(k^w*(d)), yielding O(n k^w*(d)) deadends.There are at most kn values between two sucsesive deadends: O(k n^2 k^w*(d)) number of nodes in the search space. Since at most O(2^w*(d)) constraints are check we get O(n^2 (2k)^(w*(d)+1).
- Alternatively, if we have O(n k^w*(d)) leaves, we have k to n times as many internal nodes, yielding between O(n k^(w*(d)+1))
- And O(n^2 k^w*(d)) nodes.

Analysis of backjumping and learning along DFS?

- Can we have a better bound than $O(n^2 k^m)$?

Look-back for SAT

- A partial assignment is a set of literals: sigma
- A jumpback set if a J-clause:
- Upon a leaf deadend of x resolve two clasues, one enforcing x and one enforcing ~x relative to the current assignment
- A clause forces x relative to assignment sigma if all the literals in the clause are negated in sigma.
- Resolving the two clauses we get a nogood.
- If we identify the earliest two clauses we will find the earliest condlict.
- The argument can be extended to internal deadends.

Look-back for SAT

procedure SAT-CBJ-LEARN

Input: A CNF theory φ, assigned variables σ over $x_1, ..., x_{i-1}$, unassigned variables X,

Output: Either a solution, or a decision that the network is inconsistent.

1. $J_i \leftarrow \emptyset$
2. While $1 \leq i \leq n$
3. Select the next variable: $x_i \in X$, $X \leftarrow X - \{x_i\}$
4. instantiate $x_i \leftarrow$ SELECTVALUE-CBJ.
5. If x_i is null (no value returned), then
6. add J_{x_i} to φ (learning)
7. $iprev \leftarrow$ index of last variable in J_i (*backjump*)
8. $J_i \leftarrow resolve(J_i, J_{prev})$ (merge conflict sets)
9. else,

10 $i \leftarrow i + 1$ (go forward)

11. $J_i \leftarrow \emptyset$ (reset conflict set)
12. Endwhile
13. if i = 0 Return "inconsistent"
14. else, return the set of literals σ

end procedure

subprocedure SELECTVALUE-CBJ

1. If CONSISTENT$(\sigma \cup x_i)$ then return $\sigma \leftarrow \sigma \cup \{x_i\}$
2. If CONSISTENT$(\sigma \cup \neg x_i)$ then return $\sigma \leftarrow \sigma \cup \{\neg x_i\}$
3. else,
4. determine α and β the two earliest clauses forcing x_i and $\neg x_i$,
5. $J_i \leftarrow resolve(\alpha, \beta)$.
5. Return $x_i \leftarrow$ null (no consistent value)

end procedure

Figure 6.12: Algorithm SAT-CBJ-LEARN

Integration of algorithms

procedure FC-CBJ
Input: A constraint network $\mathcal{R} = (X, D, C)$.
Output: Either a solution, or a decision that the network is inconsistent.

    $i \leftarrow 1$ (initialize variable counter)
    call SELECTVARIABLE (determine first variable)
    $D'_i \leftarrow D_i$ for $1 \le i \le n$ (copy all domains)
    $J_i \leftarrow \emptyset$ (initialize conflict set)
    while $1 \le i \le n$
        instantiate $x_i \leftarrow$ SELECTVALUE-FC-CBJ
        if x_i is null (no value was returned)
            $iprev \leftarrow i$
            $i \leftarrow$ latest index in J_i (backjump)
            $J_i \leftarrow J_i \cup J_{iprev} - \{x_i\}$
            reset each $D'_k, k > i$, to its value before x_i was last instantiated
        else
            $i \leftarrow i + 1$ (step forward)
            call SELECTVARIABLE (determine next variable)
            $D'_i \leftarrow D_i$
            $J_i \leftarrow \emptyset$
    end while
    if $i = 0$
        return "inconsistent"
    else
        return instantiated values of $\{x_1, \ldots, x_n\}$
end procedure

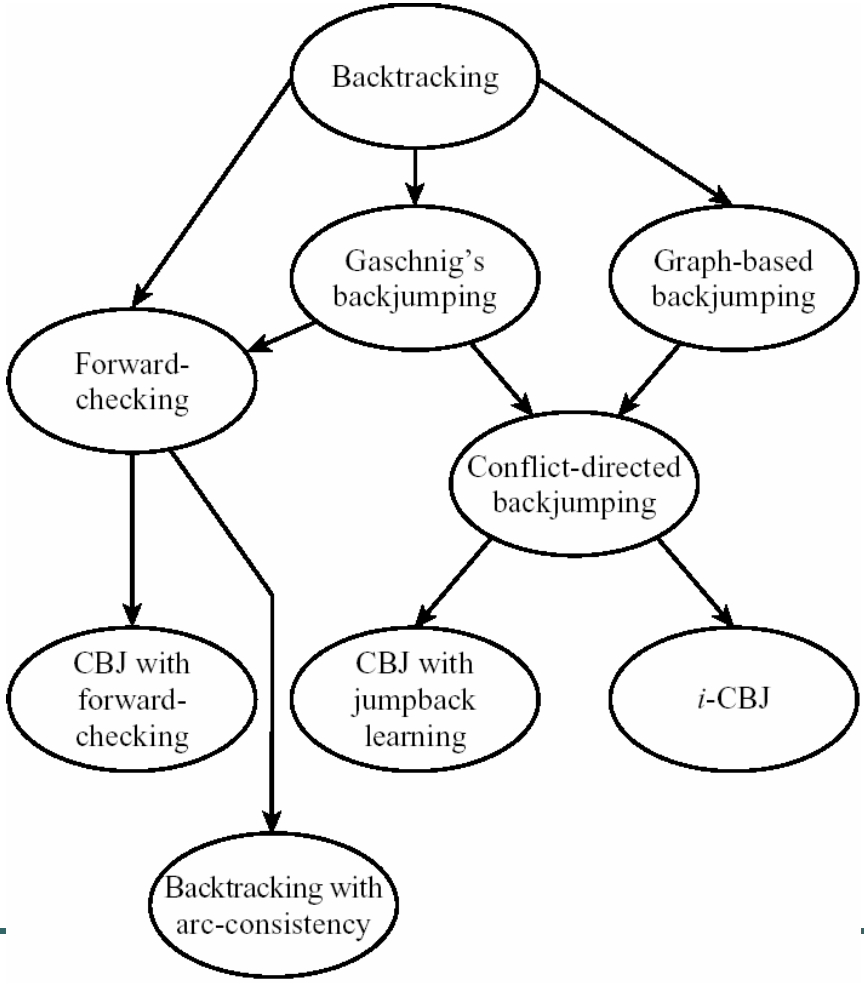
Figure 6.13: The main procedure of the FC-CBJ algorithm

```
subprocedure SELECTVALUE-FC-CBJ

    while D'_i is not empty
       select an arbitrary element a ∈ D'_i, and remove a from D'_i
       empty-domain ← false
       for all k, i < k ≤ n
          for all values b in D'_k
             if not CONSISTENT(a⃗_{i-1}, x_i = a, x_k = b)
                let R_S be the earliest constraint causing the conflict
                add the variables in R_S's scope S, but not x_k, to J_k
                remove b from D'_k
          endfor
          if D'_k is empty          (x_i = a leads to a dead-end)
             empty-domain ← true
       endfor
       if empty-domain              (don't select a)
          reset each D'_k and j_k, i < k ≤ n, to status before a was selected
       else
          return a
    end while
    return null                     (no consistent value)
end subprocedure
```

Figure 6.14: The SelectValue subprocedure for FC-CBJ.

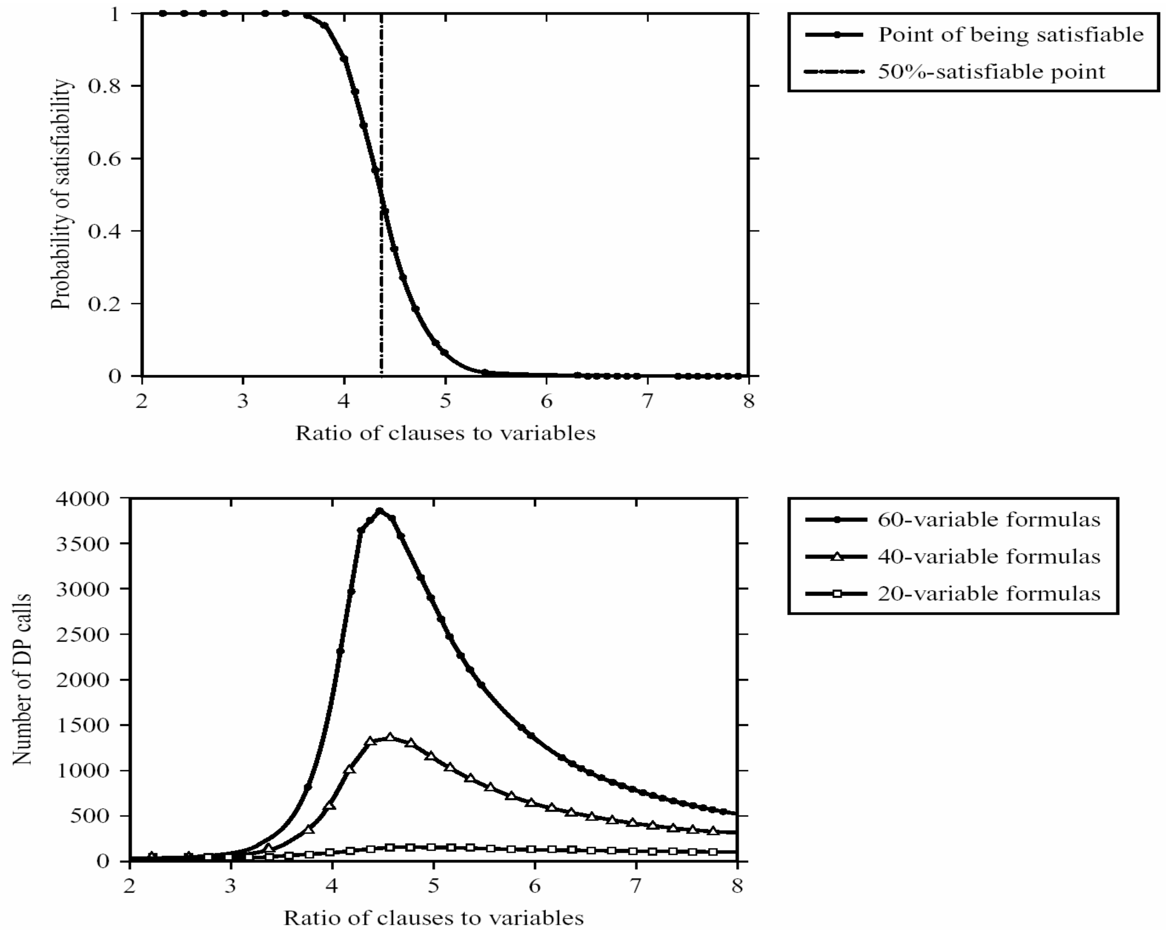
Relationships between various backtracking algrithms



Empirical comparison of algorithms

- Benchmark instances
- Random problems
- Application-based random problems
- Generating fixed length random k-sat (n,m) uniformly at random
- Generating fixed length random CSPs
- (N,K,T,C) also arity, r.

The Phase transition (m/n)



Some empirical evaluation

- Sets 1-3 reports average over 2000 instances of random csps from 50% hardness. Set 1: 200 variables, set 2: 300, Set 3: 350. All had 3 values.:
- Dimacs problems

Algorithm	Set 1		Set 2		Set 3		ssa 038		ssa 158	
FC	207	68.5		-		-	46	14.5	52	20.0
FC+AC	40	55.4	1	0.6	1	0.4	4	3.5	18	8.2
FCr-CBJ	189	69.2	222	119.3	182	140.8	40	12.2	26	10.7
FC-CBJ+LVO	167	73.8	132	86.8	119	111.8	32	11.0	8	4.5
FC-CBJ+LRN	186	63.4	32	15.6	1	0.5	23	5.5	19	8.6
FC-CBJ+LRN+LVO	160	74.0	26	14.0	1	3.8	16	3.8	13	7.1

Figure 6.16: Empirical comparison of six selected CSP algorithms. See text for explanation. In each column of numbers, the first number indicates the number of nodes in the search tree, rounded to the nearest thousand, and final 000 omitted; the second number is CPU seconds.