Some problems related to bisimilarity on BPP

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Abstract

Jančar has in 2003 shown that bisimilarity on Basic Parallel Processes (BPP) can be decided in polynomial space. Bisimilarity is studied on various subclasses of BPP as well. We present summary of known complexity bounds of some such bisimilarity problems. Moreover is here shown that deciding regularity of a BPP is *PSPACE*-complete. So far this problem was only known to be *PSPACE*-hard.

Keywords: basic parallel process, finite state system, bisimilarity, bisimulation equivalence, regularity, *PSPACE*-completeness.

1 Introduction

Equivalence checking is well studied theoretical tool for a program verification. A program and its specification are compared using some behavioral equivalence. One of the fundamental behavioral equivalences is bisimulation equivalence, also called bisimilarity. A program and a specification can be modeled using different models. We focus on well known model—basic parallel process and some of its special forms. To use equivalence checking in praxis it is crucial to know if it is possible to decide bisimilarity on used models and how complex the decision procedure is.

In the section 2 we define basic notions. In the section 3 will be summary of some known results. In section 4 we show that deciding regularity of a BPP is *PSPACE*-complete. To author's best knowledge this problem is only known to be decidable and *PSPACE*-hard. We show an algorithm working in *PSPACE* and hence combined with *PSPACE*-hardness we get *PSPACE*-completeness.

2 Basic definitions and notation

Bisimilarity (i.e., bisimulation equivalence) is defined for *labelled transition systems* (LTSs). An LTS is a tuple $(S, A, \{\xrightarrow{a}\}_{a\in A})$ where S is a (possibly infinite) set of *states*, A is a set of *actions* (or transition *labels*), and $\xrightarrow{a}\subseteq S \times S$ for each $a \in A$. We use infix notation $r \xrightarrow{a} r'$.

Given an LTS $(S, A, \{\xrightarrow{a}\}_{a\in A})$, *bisimulation equivalence* is the maximal symmetric relation B on S such that for each $(r_1, r_1') \in B$ following conditions hold:

- $\forall a, r_2 : r_1 \xrightarrow{a} r_2 \Rightarrow (\exists r_2' : r_1' \xrightarrow{a} r_2' \wedge (r_2, r_2') \in B)$
- $\forall a, r_2' : r_1' \xrightarrow{a} r_2' \Rightarrow (\exists r_2 : r_1 \xrightarrow{a} r_2 \wedge (r_2, r_2') \in B)$

States $s_1, s_2 \in S$ are *bisimilar*, written $s_1 \sim s_2$, iff there exists a bisimulation B such that $(s_1, s_2) \in B$.

A BPP can be defined as a special form of Petri net called communication-free Petri net. Concretely a BPP is a tuple $(P, \mathit{Tr}, \textsc{pre}, F, \lambda)$ where P is a finite set of *places*, Tr is a finite set of *transitions*, $\textsc{pre} : \mathit{Tr} \to P$ is a function assigning an input place to every transition, $F : (\mathit{Tr} \times P) \to \mathbb{N}$ is a function assigning output places to each transition, and $\lambda : \mathit{Tr} \to \mathcal{A}$ is a labeling function. The set of output places of the transition t we will denote by $\textsc{suc}(t) = \{p \mid F(t,p) > 0\}$.

Let $P = \{p_1, p_2, \ldots, p_k\}$ be a set of places. A *marking* is a function $M : P \to \mathbb{N}$ which assigns number of tokens to each place. Marking M can be viewed as a vector $(x_1, x_2, \ldots, x_k)$ where $x_i \in \mathbb{N}$ and $x_i = M(p_i)$. We use $\mathcal{S}_\Sigma$ to denote the set of all markings.

A transition t is *enabled* in a marking M iff $M(\textsc{pre}(t)) > 0$. Performing a transition, written $M \xrightarrow{t} M'$, means

$$M'(p) = \begin{cases} M(p) - 1 + F(t,p) & \text{if } p = \textsc{pre}(t) \\ M(p) + F(t,p) & \text{otherwise} \end{cases}$$

A BPP is called *normed*, denoted nBPP, iff from each marking we can reach an empty marking (i.e., $M(p_i) = 0$ for each $p_i \in \mathcal{S}_\Sigma$) by performing a sequence of transitions.

An LTS $(S, \mathcal{A}, \longrightarrow)$ corresponds to a BPP where $S = \mathcal{S}_\Sigma$ and $M \xrightarrow{a} M'$ iff there is some $t \in \mathit{Tr}$ such that $\lambda(t) = a$ and $M \xrightarrow{t} M'$.

A set of places $R \subseteq P$ is a *trap* iff $\forall t : \textsc{pre}(t) \in R \Rightarrow (\exists p \in R : F(t,p) \geq 1)$. A trap R is called *important* if $M \sim M'$ implies $M|_R = \mathbf{0} \Leftrightarrow M'|_R = \mathbf{0}$.

A *finite state system* (FS) is a LTS with finite set of states.

Let us have a LTS $(S, \mathcal{A}, \longrightarrow)$. We can define a distance function $\mathit{dist} : (S \times S) \to \mathbb{N}_\omega$ as follows: $\mathit{dist}(s_1, s_2) = \min(\{|w| | w \in W\} \cup \{\omega\})$, where $W = \{w \in \mathcal{A}^* \mid s_1 \xrightarrow{w} s_2\}$.

A crucial notion, introduced in [3] and used in section 4, is the notion of *DD-functions.* They are defined inductively. For every transition label a a function dd_a which, for every place s, gives the "distance to disabling" transitions with label a is a DD-function. Formally, dd_a is defined as $dd_a(s) = \min\{\, \mathit{dist}(s, s') \mid \neg\exists s'' : s' \xrightarrow{a} s'' \,\}$. Given a tuple of DD-functions $\mathcal{F} = (d_1, d_2, \ldots, d_k)$, each transition $s \xrightarrow{a} s'$ determines a *change* $\mathcal{F}(s') - \mathcal{F}(s)$, denoted δ, which is a k-tuple of values from $\{-1\} \cup \mathbb{N}_\omega \cup \{\omega\}$. For each triple $(a, \mathcal{F}, \delta)$, the function $dd_{(a,\mathcal{F},\delta)}$ (distance to disabling the action a causing the change δ of $\mathcal{F}$) is also a DD-function, defined by

$$dd_{(a,\mathcal{F},\delta)}(s) = \min\{\, \mathit{dist}(s, s') \mid \forall s'' : \text{ if } s' \xrightarrow{a} s'' \text{ then } \mathcal{F}(s'') - \mathcal{F}(s') \neq \delta \,\}.$$

All DD-functions are *bisimulation invariant*, i.e., if s and s' are bisimilar then $d(s) = d(s')$ for all DD-functions d. So equality of the values of all DD-functions is a necessary condition for two places being bisimilar. In the case of BPP this condition is also sufficient. In [3] was shown that, for any BPP, DD-functions coincide with so called 'norms':

Given $Q \subseteq \mathcal{S}_\Sigma$, we define function $\textsc{norm}_Q$ by

$$\textsc{norm}_Q(M) = \min \{ \, dist(M, M') \mid M'(p) = 0 \text{ for each } p \in Q \, \}.$$

Each $\textsc{norm}_Q$ is a linear function, i.e, for each $p \in P$ there is $c_p \in \mathbb{N}_\omega$ such that $\textsc{norm}_Q(M) = \sum_p c_p \cdot M(p)$

For all bisimulation invariant linear functions $L(M) = \sum_p c_p \cdot M(p)$ (and hence for all DD-functions) the set $R_L = \{p | c_p = \omega\}$ is an important trap.

We define relation $\preceq$ on the set of all markings $\mathcal{S}_\Sigma$ as follows. For markings $M = (x_1, x_2, \ldots, x_k)$ and $M' = (x'_1, x'_2, \ldots, x'_k)$ it holds $M \preceq M'$ iff $x_1 \leq x'_1 \wedge x_2 \leq x'_2 \wedge \ldots \wedge x_k \leq x'_k$. This relation is reflexive and transitive hence it is quasi-order. Obviously for every infinite sequence $M_1, M_2, \ldots$ of markings there exist $i < j \in \mathcal{N}$ such that $M_i \preceq M_j$ hence the relation is well-quasi-ordering. In an obvious manner is defined relation $\prec$.

3 Known results related to bisimilarity on BPP

Let us first define three known problems concerning bisimilarity on BPP.

Problem **Bisimilarity on BPP**

Instance: BPPs Δ_1, Δ_2 together with initial markings M_{I_1}, M_{I_2}

Question: Is $M_{I_1} \sim M_{I_2}$?

Problem **Bisimilarity of BPP and FS**

Instance: BPP Δ with initial marking M_I and FS Σ with initial state s_I

Question: Is $M_I \sim s_I$?

Problem **Regularity of BPP**

Instance: BPP Δ together with initial marking M_I

Question: Does a FS with initial state s_I exist such that $M_I \sim s_I$?

Similar problems can be defined for normed BPP by replacing each BPP in instances by a nBPP.

The table 1 shows best currently known complexity bounds for our three problems on BPP and nBPP. It is partially obtained from [9] and updated.

Author of this paper cooperated on two most recent results [5] and [7].

In [3] Jančar announced that his polynomial space algorithm for BPP when applied on nBPP should provide polynomial time bound of a 'reasonable' degree like $O(n^6)$. In [5] we took his algorithm and more precisely explored application on nBPP. We have presented more detailed version of the algorithm and deduced an upper bound $O(n^3)$.

	BPP	nBPP
Bisimilarity	$\in$ *PSPACE* [3] *PSPACE*-hard [10]	$\in P$ [2] ($O(n^3)$ [5]) *P*-hard [1]
Bisimilarity with FS	$\in P$ ($O(n^4)$) [7] *P*-hard [1]	$\in P$ [2] ($O(n^3)$ [5]) *P*-hard [1]
Regularity	decidable [4] *PSPACE*-hard [10]	$\in$ *NL* [8] *NL*-hard [10]

Table 1: Known complexity bounds of problems concerning bisimilarity on BPP

In [7] a general technique from [3] was taken and modified for deciding bisimilarity on BPP and FS. We presented an algorithm and explored its complexity. The upper bound of this algorithm obtained in [7] was $O(n^4)$ but it seems that analysing the algorithm more precisely may lead to $O(n^3)$. Formerly, only algorithm working in *PSPACE* was known (see [6]).

4 The regularity of a BPP is *PSPACE*-complete

As shown in table 1 regularity of BPP system is known to be decidable and *PSPACE*-hard. Using some methods from [3] we show an algorithm running in polynomial space for this problem. Hence we get that regularity of BPP is *PSPACE*-complete.

We have given a BPP $\Sigma = (P, Tr, \text{PRE}, F, \lambda)$ and initial marking M_I. The question is whether there is a FS Δ bisimilar with Σ. In the case of positive answer we will not construct existing Δ (this can be exponential to the size of Σ).

In [3] an algorithm is presented which given a BPP Σ constructs in polynomial space a mapping $\mathcal{C}_\Sigma$. Two markings M_1, M_2 of Σ are bisimilar iff $\mathcal{C}_\Sigma(M_1) = \mathcal{C}_\Sigma(M_2)$. Moreover $\mathcal{C}_\Sigma$ is n-tuple $(L_1, L_2, \ldots, L_n)$ of linear functions. Each L_i is a DD-function and in fact the norm of some set of places of Σ.

Claim 4.1 *A BPP Σ is regular iff there is finite number of mutually nonbisimilar markings.*

Proof: '$\Leftarrow$' Let the number of mutually nonbisimilar markings be finite. We can define a LTS $\Delta = (S, A, \{\xrightarrow{a}\}_{a\in A})$ where $S = \{[M]_\sim \mid M \text{ is a marking of } \Sigma\}$ and transitions are defined in obvious manner — $[M_1]_\sim \xrightarrow{a} [M_2]_\sim$ if there are markings $M_1' \in [M_1]_\sim$, $M_2' \in [M_2]_\sim$ and a transition t such that $M_1' \xrightarrow{t} M_2', \lambda(t) = a$. Then Δ is a finite state system bisimilar with Σ. It follows that Σ is regular.

'$\Rightarrow$' Now let the number of mutually nonbisimilar markings be infinite. Now we can not construct a FS which has one state for each equivalence class on markings. If two markings are nonbisimilar they can not be both bisimilar with the same state of finite state system. Hence there is not any FS bisimilar with Σ and Σ is not regular. $\Box$

Because $\mathcal{C}_\Sigma(M_1) = \mathcal{C}_\Sigma(M_2)$ for $M_1 \sim M_2$ system is regular iff we have a finite number of different possible values of $\mathcal{C}_\Sigma$ on reachable markings of Σ. An infinite number of values of $\mathcal{C}_\Sigma$ is possible iff at least one of the functions L_i has an infinite number of the possible values.

Claim 4.2 *Norm function L has infinite number of different values on markings of Σ iff there are two sequences of places $p_0, p_1, \ldots p_{n-1}$ and $p'_0, p'_1, \ldots p'_{m-1}$ for $n, m \in \mathbb{N}, n \geq 1, m \geq 1$ such that following conditions hold:*

1. $p_i \in (P \backslash R_L)$ *for* $0 \leq i < n$ *and* $p'_k \in (P \backslash R_L)$ *for* $0 \leq k < m$
2. $p_i \neq p_j$ *for* $i \neq j,\ \ 0 \leq i < n, 0 \leq j < n$ *and* $p'_k \neq p'_l$ *for* $k \neq l,\ \ 0 \leq k < m, 0 \leq l < m$
3. *for each* $p_i, 0 \leq i < n$, *there is* $t_i \in Tr$ *such that* $\text{PRE}(t_i) = p_i, p_{(i+1) mod\ n} \in \text{SUC}(t_i)$ *and for each* $p'_k, 0 \leq k < m-1$, *there is* $t'_k \in Tr$ *such that* $\text{PRE}(t'_k) = p'_k, p'_{k+1} \in \text{SUC}(t'_k)$
4. $(\cup_{i=0}^{n-1} \text{SUC}(t_i)) \cap R_L = \emptyset$ *and* $(\cup_{k=1}^{m-1} \text{SUC}(t'_k)) \cap R_L = \emptyset$
5. *for some* t_i *it holds* $|\text{SUC}(t_i) \backslash R| > 1, p'_1 \in \text{SUC}(t_i)$ *or* $F(t_i, p_{(i+1) mod\ n}) > 1, p'_1 = p_{(i+1) mod\ n}$
6. $0 < c_{p'_m} < \omega$
7. *there is a marking* M *such that* $M_I \xrightarrow{w} M$ *for some* $w \in A^*$, $M|_{R_L} = \boldsymbol{0}$ *and* $M|_{\{p_0, \ldots, p_{n-1}\}} \neq \boldsymbol{0}$

Proof: '$\Rightarrow$' We suppose a BPP Σ such that there is an infinite number of different values of function L on reachable markings. L is a linear function, i.e, for each $p \in P$ there is $c_p \in \mathbb{N}_\omega$ such that $L(M) = \sum_p c_p \cdot M(p)$.

Let's assume that there is not any reachable marking M_1 from which we can reach a marking M_2 such that $M_1 \prec M_2$, $L(M_1) < L(M_2)$ and important trap is not marked. It means that from each marking M'_1 we can reach only

- incomparable marking
- strictly smaller marking
- the same marking
- a marking M'_2 such that $M'_1 \prec M'_2$ and $L(M_1) \geq L(M'_2)$
- marking with token in an important trap

Given a marking M, there is only finite number of reachable incomparable and strictly smaller markings. This easily follows from the fact that the relation $\preceq$ is well-quasi-ordering on the set of all markings, from properties of a BPP and from our assumption. Because we can reach only finite number of markings, we have finite number of different values of function L. But we can reach also markings greater than M_1. The value of function L on such markings is smaller or equal $L(M_1)$ and there is only finite number of values smaller or equal $L(M_1)$. There is only one value of L for all markings with tokens in important trap. Hence we get contradiction with the fact that we can reach infinite number of markings with different value of L.

It follows that, for Σ, a reachable marking M_1 exist from which we can reach a marking M_2 such that $M_1 \prec M_2$, $L(M_1) < L(M_2)$ and an important trap is not marked. The sequence of transitions leading from M_1 to M_2 can be repeated infinite times generating greater and greater markings. This means that in BPP is something like cycle which can get a token and generate infinite number of tokens without marking an important trap. This is described in conditions of claim 4.2.

'$\Leftarrow$' Places $p_0, \ldots, p_{n-1}$ together with transitions $t_0, \ldots, t_{n-1}$ related in point 3 form a 'cycle' which repeating causes generation of tokens. We consider only cycles containing each place only once (point 2). Other cycles can be divided into smaller ones. There is at least one transition in this cycle which has more than one output place or gives more then one token to its output place (point 5). This transition ensures generation of at least one token in each repetition of the cycle.

There are sequences of places $(p'_0, p'_1, \ldots p'_{m-1})$ and transitions between them $(t'_0, \ldots t'_{m-1}$ in point 3) which transports generated tokens into some place (r_{m-1}) with finite positive coefficient of function L (point 6). Point 4 ensures that an important trap can not be marked. From the point 7 follows that there is a possibility to get a token to some place of our cycle from initial marking without marking an important trap.

Hence we can generate infinite number of token into place with positive finite constant in function L. This means that we can reach infinite number of markings with different values of function L. $\square$

Theorem 4.3 *The regularity of a BPP is in PSPACE and hence is PSPACE-complete.*

Proof: Using algorithms from [3] we can compute $\mathcal{C}_\Sigma = (L_1, L_2, \ldots, L_n)$ and important traps in polynomial space.

We have finite number of places and transitions. We can check all possible subsets of places if they are in the cycle according the conditions from claim 4.2 and in the case of positive answer we can check existence of sequence transporting tokens to some place with finite coefficient on some of functions L_i. This can be done obviously in polynomial space (it is even in *NP*). If a desired cycle and a sequence are found, the BPP system is not regular. In the other case the system is regular. $\square$

5 Future work

As the table 1 together with section 4 suggests there are known quite proper complexity bounds for all problems concerning bisimilarity on BPP and its subclasses. But many problems are open in the case of so called weak bisimilarity where we allow silent actions.

It is not known whether weak bisimilarity on BPP and even on nBPP is decidable or not. Jančar in [3] remarked that his method can be useful in the case of weak bisimilarity. But this is not so straightforward and we will work on it.

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