Examples of proofs

I) $\exists x \forall y\, P(x,y) \supset \forall y \exists x\, P(x,y)$

***a) Proof by natural deduction*:**

1. $\exists x \forall y\, P(x,y)$	assumption
2. $\forall y\, P(a,y)$	E∃ (1)
3. $P(a,y)$	E∀ (2)
4. $\exists x\, P(x,y)$	I∃ (3)
5. $\forall y \exists x\, P(x,y)$	I∀ (4)

Note: this proof is valid, because we have to eliminate the existential quantifier first. The inverse implication $\forall y \exists x\, P(x,y) \supset \exists x \forall y\, P(x,y)$ is **not** a logically valid formula, because

1. $\forall y \exists x\, P(x,y)$	assumption
2. $\forall y\, P(f(y),y)$	E∃ (1) – the variable x is in the scope of the general quantifier!
3. $P(f(y),y)$	E∀ (2)

Now there is no reasonable way to continue.

***b) Proof by resolution method*:**

$\exists x \forall y\, P(x,y) \wedge \exists y \forall x\, \neg P(x,y)$	negated formula (A)
$\forall y\, P(a,y) \wedge \forall z\, \neg P(z,b)$	Skolemization and renaming variable y (the second)
$\forall y \forall z\, P(a,y) \wedge \neg P(z,b)$	quantifiers to the left

1. $P(a,y)$	
2. $\neg P(z,b)$	
3. #	empty clause by unification: a/z, b/y

inverse implication: $\forall y \exists x\, P(x,y) \supset \exists x \forall y\, P(x,y)$

$\forall y \exists x\, P(x,y) \wedge \forall x \exists y\, \neg P(x,y)$ negated (B)

$\forall y\, P(f(y),y) \wedge \forall x\, \neg P(x,g(x))$ Skolemization (now x,y are in the scope of ∀)

1. $P(f(y),y)$
2. $\neg P(x,g(x))$
3. $\neg P(f(y),g(f(y)))$ $f(y)/x$ into 2. (in the aim to unify 1 and 2)
4. $P(f(g(f(y))), g(f(y)))$

….

no way to unify the two clauses, they are ***not unifiable***, the formula is ***not a tautology***

(II) $\exists x\, [P(x) \wedge Q(x)] \supset [\exists x\, P(x) \wedge \exists x\, Q(x)]$

a) Proof by natural deduction

1. $\exists x\, [P(x) \wedge Q(x)]$	assumption
2. $[P(a) \wedge Q(a)]$	E$\exists$ (1)
3. $P(a)$	E$\wedge$ (2)
4. $Q(a)$	E$\wedge$ (2)
5. $\exists x\, P(x)$	I$\exists$ (3)
6. $\exists x\, Q(x)$	I$\exists$ (4)
7. $[\exists x\, P(x) \wedge \exists x\, Q(x)]$	I$\wedge$ (5,6)

b) Proof by resolution method

First, negate the formula:

$\exists x\, [P(x) \wedge Q(x)] \wedge \neg[\exists x\, P(x) \wedge \exists x\, Q(x)] \Leftrightarrow$

$\exists x\, [P(x) \wedge Q(x)] \wedge [\forall x\, \neg P(x) \vee \forall x\, \neg Q(x)]$

Transform the negated formula *into Skolem clausal form*:

Eliminate $\exists$ and rename x

$P(a) \wedge Q(a) \wedge [\forall x\, \neg P(x) \vee \forall y\, \neg Q(y)] \Leftrightarrow$ ($\forall$ to the left)

$\forall x \forall y\, [P(a) \wedge Q(a) \wedge [\neg P(x) \vee \neg Q(y)]]$

1.	$P(a)$	
2.	$Q(a)$	
3.	$\neg P(x) \vee \neg Q(y)$	
4.	$\neg Q(a)$	resolution 1., 3., a/x
5.	#	contradiction 2. and 4.

Again, the inverse implication is ***not*** valid: **$[\exists x\, P(x) \wedge \exists x\, Q(x)] \supset \exists x\, [P(x) \wedge Q(x)]$**

***Natural deduction*:**

1. $\exists x\, P(x) \wedge \exists x\, Q(x)$	assumption
2. $\exists x\, P(x)$	E$\wedge$ (1)
3. $\exists x\, Q(x)$	E$\wedge$ (1)
4. $P(\mathbf{a})$	E$\exists$ (2)
5. $Q(\mathbf{b})$	E$\exists$ (3) **-** we ***must*** use a ***different konstant!***
6. $P(a) \wedge Q(b)$	I$\wedge$ (4,5)
7. $\exists x\, P(x) \wedge \exists y\, Q(y)$	I$\exists$ (6)

No way to prove **$\exists x\, [P(x) \wedge Q(x)]$.**

Resolution method:

Negation : $\exists x\, P(x) \wedge \exists x\, Q(x) \wedge \forall x\, [\neg P(x) \vee \neg Q(x)]$

Skolemization and clauses:

1.	$P(a)$	
2.	$Q(b)$	
3.	$\neg P(x) \vee \neg Q(x)$	
4.	$\neg Q(a)$	resolution 1, 3, a/x

No way to continue …

(III) $[\forall x\, P(x) \vee \forall x\, Q(x)] \supset \forall x\, [P(x) \vee Q(x)]$

a) Proof by ***natural deduction***:

1. $\forall x\, P(x) \vee \forall x\, Q(x)$ assumption
   - 2.1. $\forall x\, P(x)$ hypotheses of a branching proof
   - 2.2. $P(x)$ E$\forall$ (2.1)
   - 2.3. $P(x) \vee Q(x)$ I$\vee$ (2.2)
   - 2.4. $\forall x\, (P(x) \vee Q(x))$ I$\forall$ (2.3)
2. $\forall x\, P(x) \supset \forall x\, (P(x) \vee Q(x))$
   - 3.1. $\forall x\, Q(x)$ hypotheses of a branching proof
   - 3.2. $Q(x)$ E$\forall$ (3.1)
   - 3.3. $P(x) \vee Q(x)$ I$\vee$ (3.2)
   - 3.4. $\forall x\, (P(x) \vee Q(x))$ I$\forall$ (3.3)
3. $\forall x\, Q(x) \supset \forall x\, (P(x) \vee Q(x))$
4. $[\forall x\, P(x) \supset \forall x\, (P(x) \vee Q(x))] \wedge [\forall x\, Q(x) \supset \forall x\, (P(x) \vee Q(x))]$ I$\wedge$ (2,3)
5. (4) $\supset [[\forall x\, P(x) \vee \forall x\, Q(x)] \supset \forall x\, (P(x) \vee Q(x))]$ Theorem
6. $[\forall x\, P(x) \vee \forall x\, Q(x)] \supset \forall x\, (P(x) \vee Q(x))$ MP (4,5)
7. $\forall x\, (P(x) \vee Q(x))$ MP (1,6)

The steps 4 – 6 are usually omitted, because we have proven them earlier.

b) Proof by ***resolution method***; first, ***negate the formula***

$\neg[\forall x\, P(x) \vee \forall x\, Q(x)] \supset \forall x\, [P(x) \vee Q(x)] \Leftrightarrow [\forall x\, P(x) \vee \forall x\, Q(x)] \wedge \exists x\, [\neg P(x) \wedge \neg Q(x)]$

Skolemization: $[\forall x\, P(x) \vee \forall x\, Q(x)] \wedge [\neg P(a) \wedge \neg Q(a)]$

1.	$P(x) \vee Q(x)$	
2.	$\neg P(a)$	
3.	$\neg Q(a)$	
4.	$Q(a)$	resolution 1, 2, a/x
5.	#	contradiction 3 and 4

(IV) $\exists x\, P(x) \supset (\forall x\, [P(x) \supset Q(x)] \supset \exists x\, Q(x))$

a) *Proof by* ***natural deduction***:

1. $\exists x\, P(x)$	assumption 1
2. $\forall x\, [P(x) \supset Q(x)]$	assumption 2
3. $P(a)$	E$\exists$ (1)
4. $P(a) \supset Q(a)$	E$\forall$ (2)
5. $Q(a)$	MP (3,4)
6. $\exists x\, Q(x)$	I$\exists$ (5)

Comments: we first eliminate existential quantifier by substituting *a* for *x* (step 3). Then we eliminate general quantifier by substituting *a* for *x* (step 4), because we can substitute *any* term for a generally quantified variable ("what is valid for everybody is also valid for somebody").

b) *Proof by* ***resolution method***:

– First, negate the formula: $\neg\{\exists x\, P(x) \supset (\forall x\, [P(x) \supset Q(x)] \supset \exists x\, Q(x))\} \Leftrightarrow$
  $\exists x\, P(x) \wedge \forall x\, [P(x) \supset Q(x)] \wedge \forall x\, \neg Q(x)$
– Eliminate $\exists$ (Skolemisation), rename the second *x*, and $\forall$s to the left:
  $\forall x \forall y\, \{P(a) \wedge [P(x) \supset Q(x)] \wedge \neg Q(y)\}$
– Clauses
  1. $P(a)$
  2. $\neg P(x) \vee Q(x)$
  3. $\neg Q(y)$
  4. $Q(a)$ resolution 1, 2, *a/x*
  5. # resolution 3, 4, *a/y*